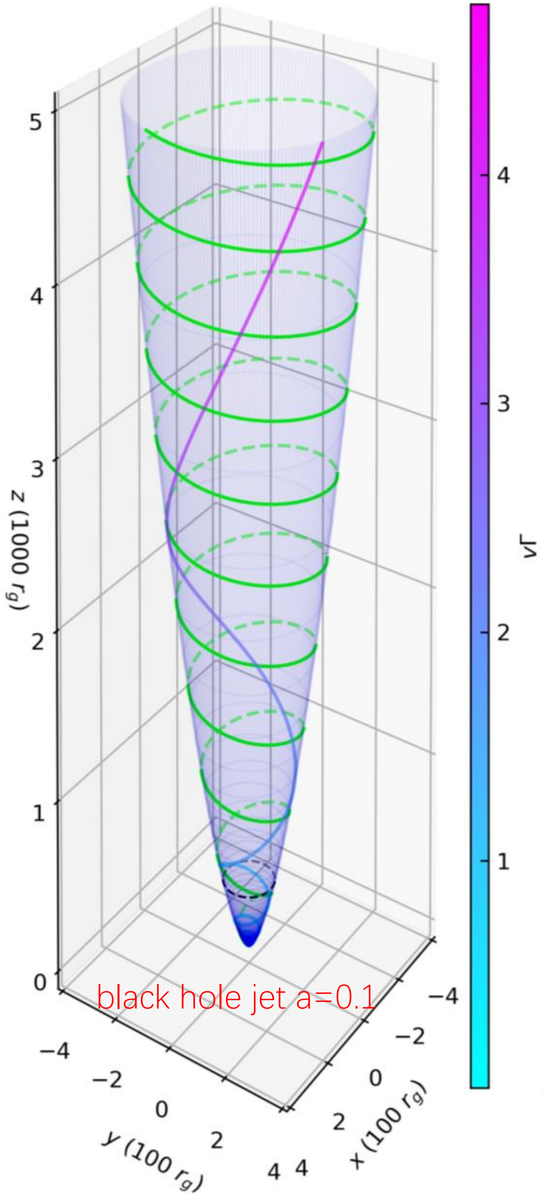

A story about "making" a jet or a lightsaber.

Magnetic Field is the secret behind the mysterious power of the universe.

Chen & Zhang 2021, ApJ, 906, 105

Analytical Solution of Magnetically Dominated Jet/Wind:

jet launching, acceleration and collimation

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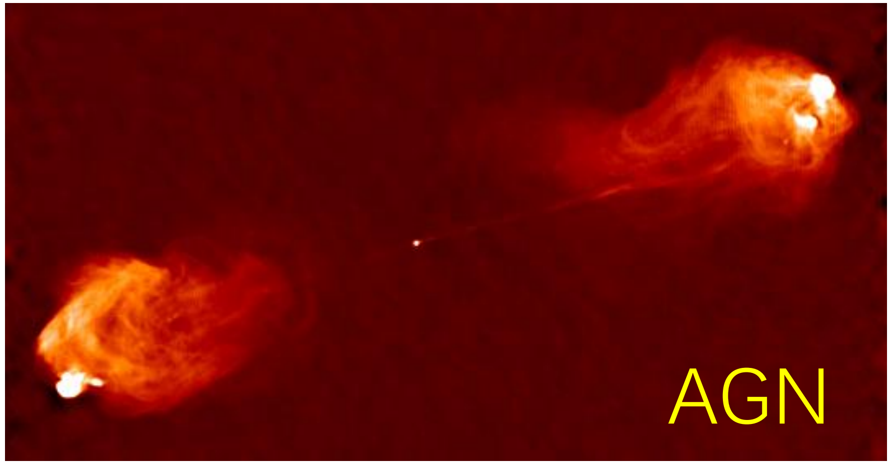
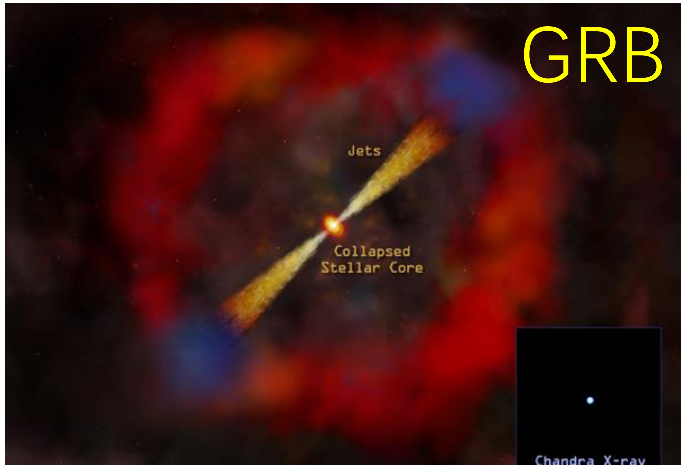
collaborator: Bing Zhang

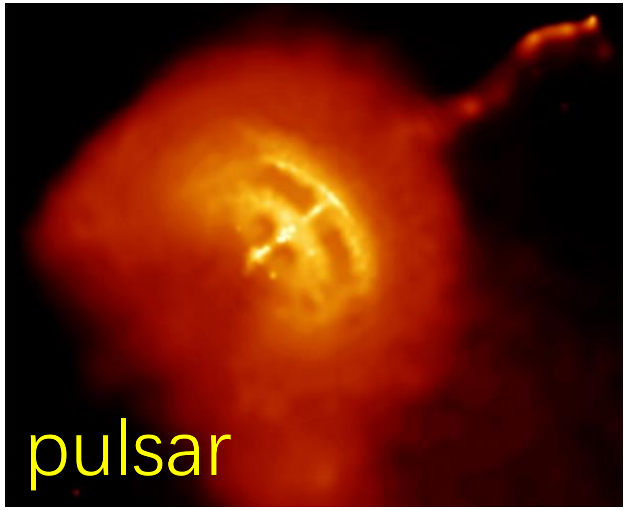
@ TSINGHUA (2021.03.25)

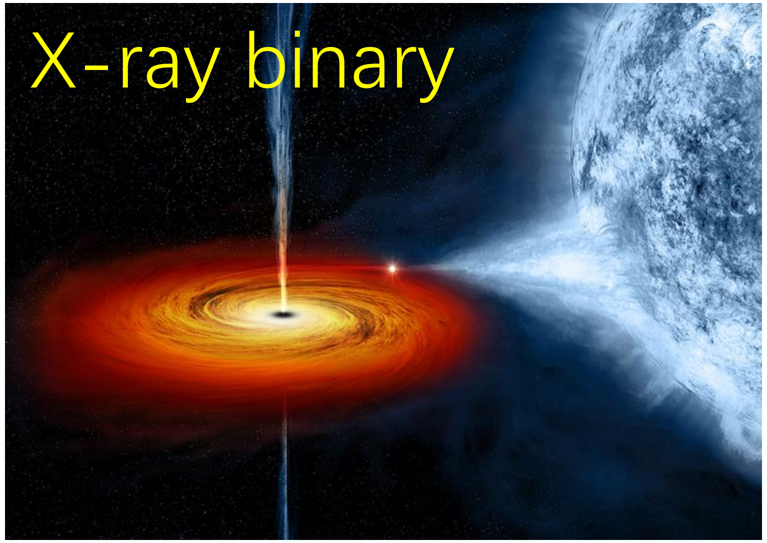
Chen & Zhang 2021, ApJ, 906, 105

Astrophysical jets



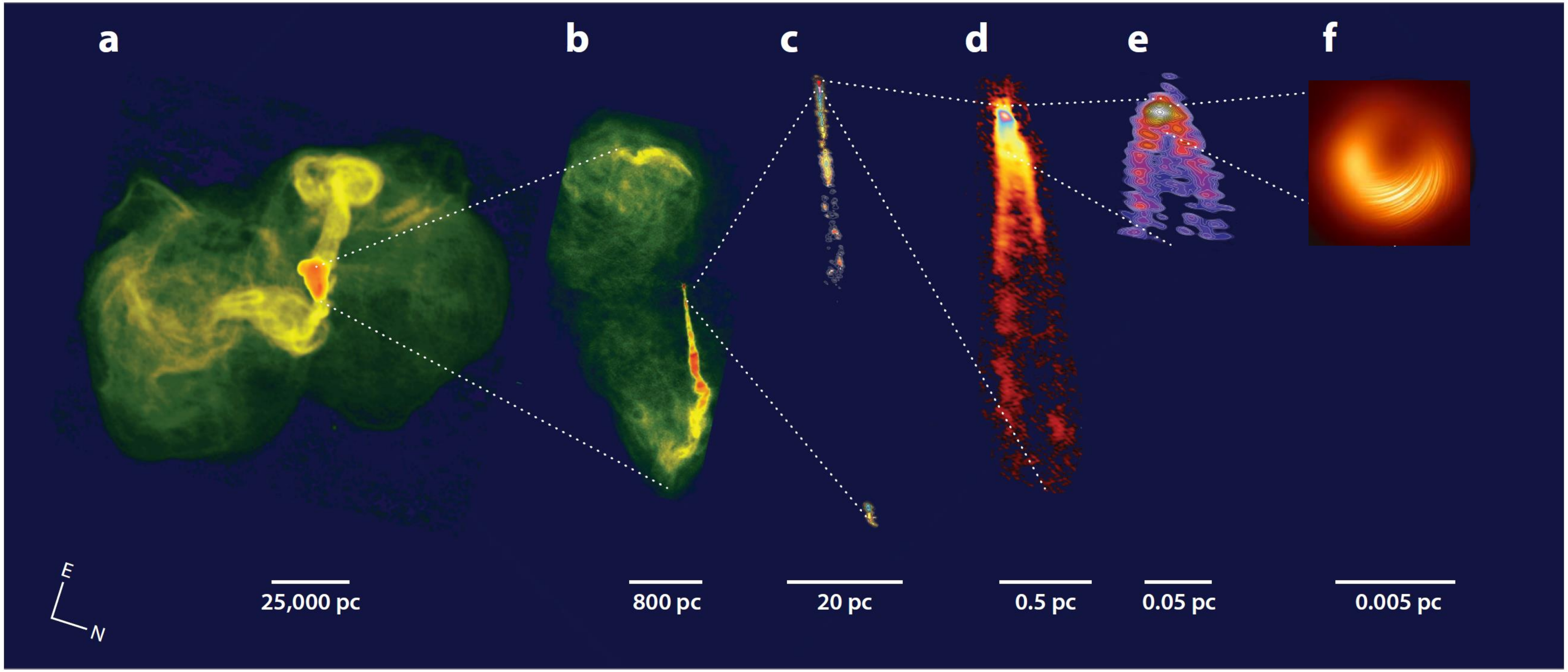






Astrophysical jets – M87

collimation : ~10^6 (~10^10)



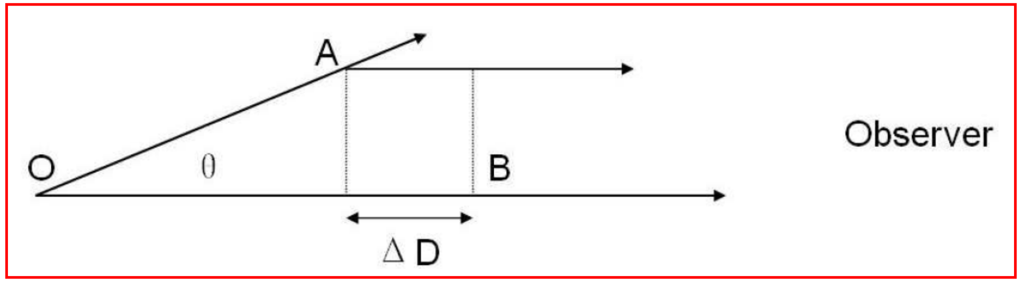
Blandford et al. 2019; EHT Collaboration 2021

Astrophysical jets

- superluminal motion

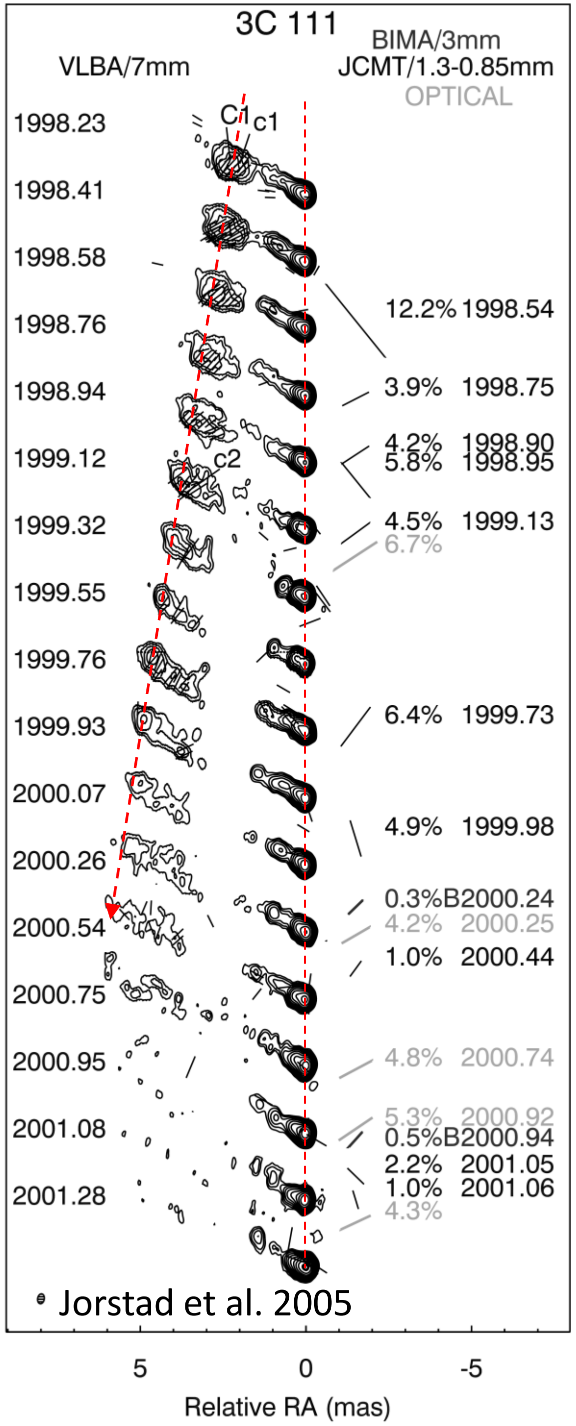
$$v_{\text{app}} \gg c$$

- geometrical effect → relativistic



$$\Delta D/c = \Delta t - \beta \Delta t \cos\theta$$

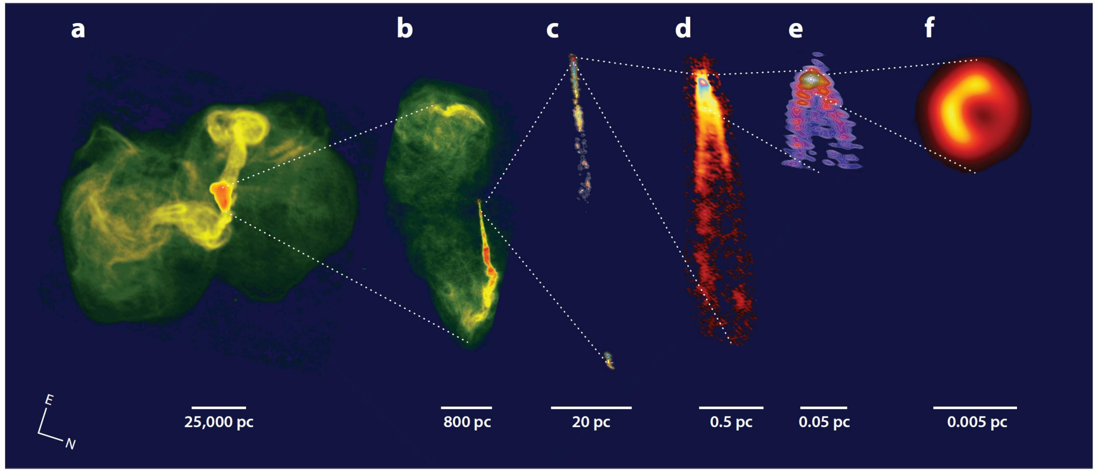
$$\boxed{\beta_{app} \equiv \frac{v_{app}}{c} = \frac{1}{c}\frac{OA\sin\theta}{\Delta t - \beta\Delta t\cos\theta} = \frac{\beta\sin\theta}{1-\beta\cos\theta}}$$

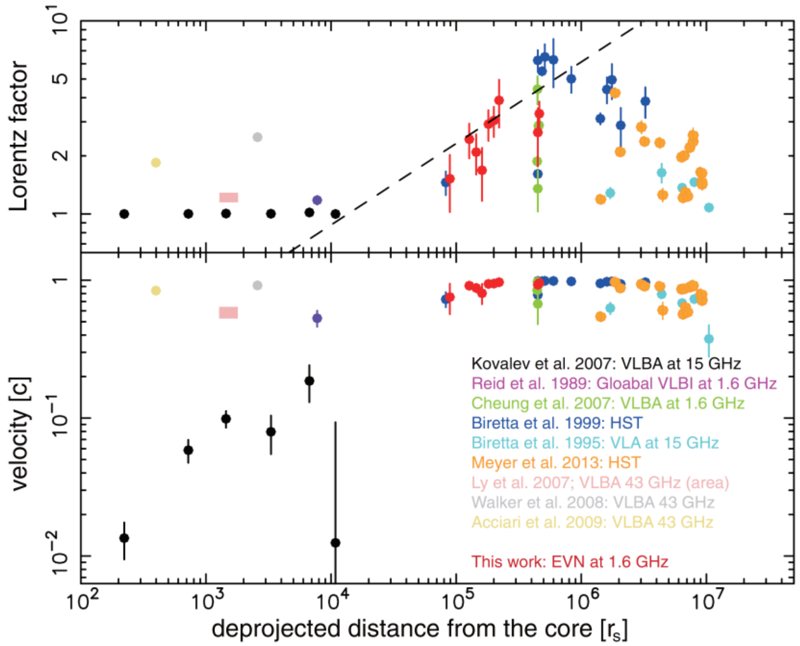


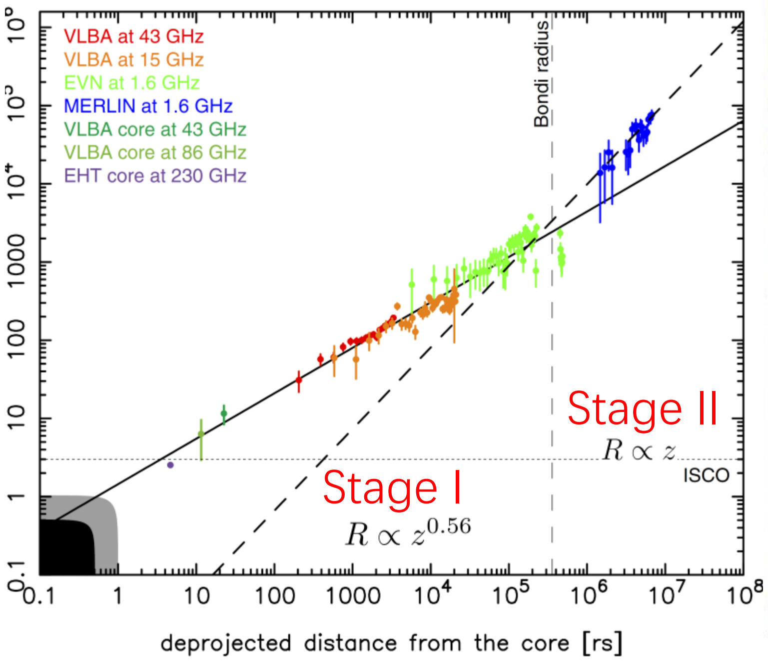
- Jorstad et al. 2005

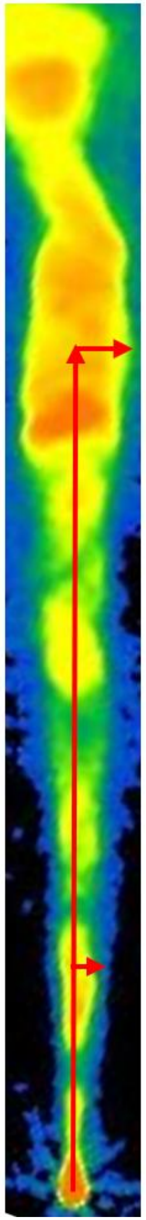
Astrophysical jets

- Stage I:
  - collimating
  - accelerating
  - “quasi-parabolic”
- Stage II:
  - “collimating”
  - “conical”
- “Stage III”
  - terminal, lobe









Astrophysical jets

- Relativistic ($\Gamma \sim 100$)
- Collimation ($\theta \sim 1^{o}$)
- Propagating scale many order of magnitude ($\sim 10^{10}$ Rg)

- Variability
- Broadband wavelength emissions
- Polarization
- …

Astrophysical jets: problems

- Jet launching, acceleration, collimation
- Energy dissipation and particle acceleration
- Radiative processes vs. observations
- …

Astrophysical jets: problems

- Jet launching, acceleration, collimation
- Energy dissipation and particle acceleration
- Radiative processes vs. observations
- …

Astrophysical jets: problems

- Magnetic centrifugal force
- Magnetic pressure gradient force
- Pinch force

magnetically driven (BZ/BP)

- Jet launching, acceleration and collimation

  numerical versus quantitative analytical

- Explain observations of jets (e.g., VLBI data)

Astrophysical jets: magnetically driven jet

- Magnetosphere

  stellar wind (Chandrasekhar,⋯), pulsar (Goldreich & Julian,⋯), MHD

- Numerical

  simulation (Narayan, Yuan, Tchekhovskoy, McKinney, Mizuno, Bai,⋯), semi-analysis (Blandford, Narayan, Spruit, Cao, Contopoulos, Huang,⋯)

- Analytical
- Type I: assume magnetic field configuration (Spruit, Cao,⋯)
- Type II: asymptotic properties (Beskin, Lyubarsky, Komissarov, Narayan, Tchekhovskoy,⋯)
- Type III: monopolar, cylindrical, parabolic (Michel, Blandford,⋯)

A global jet theory should satisfy

- physically (mathematically) reasonable
- cover from non-relativistic to relativistic regime
- match observations and previous theoretical results
- explicitly analytical and comprehensive
- can be approximate, but should be accurate enough

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- physically (mathematically) reasonable
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Chen & Zhang 2021, ApJ, 906, 105

Content

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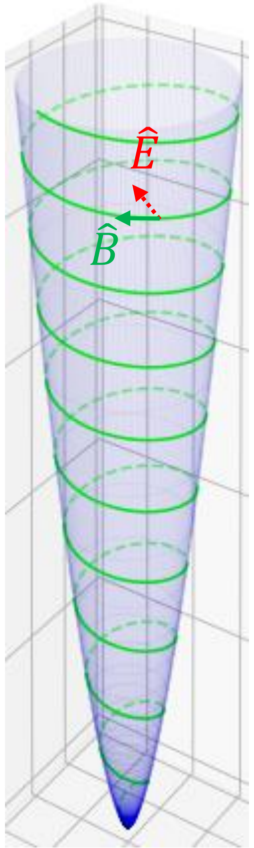
S2: the governing equation: axisymmetric, steady, no-GR



$$\mathbf{B} = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}\hat{\mathbf{r}} - \frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}\hat{\theta} + \frac{\Phi}{r\sin\theta}\hat{\phi}$$

$$\boxed{\Psi} = r\sin\theta A_\phi \qquad \boxed{\Phi} = r\sin\theta B_\phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\boxed{\Omega}\nabla\Psi = -\Omega r \sin\theta\hat{\phi} \times \mathbf{B}$$

$$\frac{\mathbf{j}}{\sigma_{\rm c}} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \overset{\text{ideal MHD}}{=\!=\!=\!=} 0$$

$$\rho\left(\mathbf{u}\cdot\bigtriangledown\right)\mathbf{u} = \rho_{\rm e}\mathbf{E} + \mathbf{j}\times\mathbf{B} \overset{\text{force free}}{=\!=\!=\!=} 0$$

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}$$
$$+\Phi'\Phi - \left\{\frac{\Omega'}{\Omega}\left[\left(\frac{\partial\Psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2\right] + \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}\right\}\left(\Omega r\sin\theta\right)^2 = 0$$

the “pulsar” equation (established 1960s)

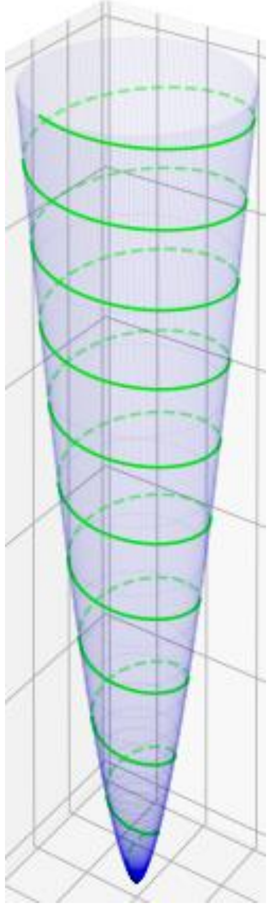
How does nature work? (I)

- Simulation: BH rotating slow or fast produce similar jet configuration
  (Tchekhovskoy, McKinney & Narayan 2008)

- by chance? physical? implication?
  (twisted magnetic field, velocity)

- If real, rotating to be very small (~vanish), similar configuration

- Math expect: two terms (equations): non-rotating and rotating



How does nature work? (II)

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}$$
$$+\Phi'\Phi - \left\{\frac{\Omega'}{\Omega}\left[\left(\frac{\partial\Psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2\right] + \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}\right\}(\Omega r\sin\theta)^2 = 0$$

if real

non-rotating $$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta} = 0$$

rotating $$\frac{\Phi'\Phi}{\Omega^2 r^2\sin^2\theta} - \left\{\frac{\Omega'}{\Omega}\left[\left(\frac{\partial\Psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2\right] + \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}\right\} = 0$$

SEEMS STRANGE! do not have to be equal to 0 simultaneously!

How does nature work? (III)

- Unless the nature works in a "subtle" way
- No expectation
- A surprise

non-rotating	$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}=0$
rotating	$\frac{\Phi'\Phi}{\Omega^2 r^2\sin^2\theta}-\left\{\frac{\Omega'}{\Omega}\left[\left(\frac{\partial\Psi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2\right]+\frac{\partial^2\Psi}{\partial r^2}+\frac{2}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}\right\}=0$

- two solutions match each other!
- both are analytical!

S3: an approximate solution: rotating term

- Ψ, magnetic flux, the natural boundary: vanish at $\theta = 0$
- angular velocity, the ansatz ($\lambda = 0$ threading CO; $\lambda < 0$ threading AD)

$$\Omega = \alpha \Psi^{\lambda}$$

- Φ , the enclosed current ($\beta = 2$)

$$\Phi = -\beta \Omega \Psi$$

$$\boxed{\frac{\Phi' \Phi}{\Omega^2 r^2 \sin^2 \theta} - \left\{ \frac{\Omega'}{\Omega} \left[\left(\frac{\partial \Psi}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial \Psi}{\partial \theta} \right)^2 \right] + \frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial \Psi}{\partial \theta} \right\} = 0}$$

$$\Big\Downarrow \; \Psi = H_{\mathrm{r}}(r) \, T_{\mathrm{r}}(\theta)$$

$$\boxed{\lambda r^2 \left(\frac{H_{\mathrm{r}}'}{H_{\mathrm{r}}} \right)^2 + r^2 \frac{H_{\mathrm{r}}''}{H_{\mathrm{r}}} + 2r \frac{H_{\mathrm{r}}'}{H_{\mathrm{r}}} = \frac{\beta^2 (1 + \lambda)}{\sin^2 \theta} - \lambda \left(\frac{T_{\mathrm{r}}'}{T_{\mathrm{r}}} \right)^2 - \frac{T_{\mathrm{r}}''}{T_{\mathrm{r}}} - \cot \theta \frac{T_{\mathrm{r}}'}{T_{\mathrm{r}}}}$$

S3: an approximate solution: rotating term

$$\boxed{\lambda r^2\left(\frac{H_\mathrm{r}'}{H_\mathrm{r}}\right)^2+r^2\frac{H_\mathrm{r}''}{H_\mathrm{r}}+2r\frac{H_\mathrm{r}'}{H_\mathrm{r}}=\frac{\beta^2(1+\lambda)}{\sin^2\theta}-\lambda\left(\frac{T_\mathrm{r}'}{T_\mathrm{r}}\right)^2-\frac{T_\mathrm{r}''}{T_\mathrm{r}}-\cot\theta\frac{T_\mathrm{r}'}{T_\mathrm{r}}=(1+\lambda)\nu^2+\nu}$$

- r component solution $H_\mathrm{r}(r)=r^\nu$
- θ component ($y=\sin^2\theta$, spherical/cylindrical coordinate)

$$4(1-y)y^2T_\mathrm{r}T_\mathrm{r}''+4\lambda(1-y)y^2T_\mathrm{r}'^2-2(3y-2)yT_\mathrm{r}T_\mathrm{r}'-\beta^2(\lambda+1)T_\mathrm{r}^2+\nu(\nu+1+\lambda\nu)yT_\mathrm{r}^2=0$$

- at $\theta\ll 1$: $T_\mathrm{r}(\theta)\propto\theta^\beta$

$\beta=2$

$$\beta(\beta-2)-\left[\nu(1+\nu+\lambda\nu)-\frac{\beta^2}{3}(1+\lambda)-\frac{\beta}{3}+4a_2(1+\beta+\beta\lambda)\right]\Omega^2r^2\theta^4\approx 0$$

magnetic flux conservation

S3: an approximate solution: rotating term

- θ component ($s \equiv \nu\lambda$, $4(1-y)y^2 T_\mathrm{r} T_\mathrm{r}'' + 4\lambda(1-y)y^2 T_\mathrm{r}'^2 - 2(3y-2)yT_\mathrm{r}T_\mathrm{r}' - \beta^2(\lambda+1)T_\mathrm{r}^2 + \nu(\nu+1+\lambda\nu)yT_\mathrm{r}^2 = 0$)

$$T_\mathrm{r}(y) = A_2 e^{\frac{\nu}{s+\nu}\int_1^y \frac{G_1(t)+A_1G_2(t)}{A_1G_3(t)+G_4(t)}dt}$$

$$a_1 = \frac{b}{2} - \frac{s}{2} + \frac{\beta s}{2\nu} - \frac{\nu}{2};\ b_1 = \frac{1}{2} + \frac{b}{2} + \frac{s}{2} + \frac{\beta s}{2\nu} + \frac{\nu}{2};\ c_1 = 1 + \beta + \frac{s\beta}{\nu},$$
$$a_2 = -\frac{\beta}{2} - \frac{s}{2} - \frac{\beta s}{2\nu} - \frac{\nu}{2};\ b_2 = \frac{1}{2} - \frac{b}{2} + \frac{s}{2} - \frac{\beta s}{2\nu} + \frac{\nu}{2};\ c_2 = 1 - b - \frac{\beta s}{\nu},$$
$$G_1 = \frac{\beta(s+\nu)}{2\nu}\ {}_2F_1(a_1,b_1,c_1,t)\,t^{-1+\frac{\beta(s+\nu)}{2\nu}} + \frac{a_1b_1}{c_1}\ {}_2F_1(a_1+1,b_1+1,c_1+1,t)\,t^{\frac{\beta(s+\nu)}{2\nu}},$$
$$G_2 = \frac{-\beta(s+\nu)}{2\nu}\ {}_2F_1(a_2,b_2,c_2,t)\,t^{-1-\frac{\beta(s+\nu)}{2\nu}} + \frac{a_2b_2}{c_2}\ {}_2F_1(a_2+1,b_2+1,c_2+1,t)\,t^{\frac{-\beta(s+\nu)}{2\nu}},$$
$$G_3 = {}_2F_1(a_2,b_2,c_2,t)\,t^{\frac{-\beta(s+\nu)}{2\nu}},$$
$$G_4 = {}_2F_1(a_1,b_1,c_1,t)\,t^{\frac{\beta(s+\nu)}{2\nu}},$$

Hypergeometric function

S3: an approximate solution: non-rotating term

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}=0$$

- solution $\boxed{\Psi = r^{\nu}T_{\rm nr}\left(\theta\right)}$

$$\boxed{T_{\rm nr}\left(\theta\right)=C_2 y\ {}_2F_1\left(1-\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2},2,y\right)\ =\ {}_2F_1\left(\frac{\nu}{2}-\frac{1}{2},-\frac{\nu}{2},\frac{1}{2},\mu^2\right)-C_1\mu\ {}_2F_1\left(\frac{1}{2}-\frac{\nu}{2},\frac{\nu}{2},\frac{3}{2},\mu^2\right)}$$

- at $\theta \ll 1$: $T_{\rm nr} \propto \theta^2$

$$\mu=\cos\theta,$$
$$C_1=\frac{\nu\Gamma\left(3/2-\nu/2\right)\Gamma\left(\nu/2\right)}{\Gamma\left(1-\nu/2\right)\Gamma\left(\nu/2+1/2\right)},$$
$$C_2=\frac{\Gamma\left(3/2-\nu/2\right)\Gamma\left(1+\nu/2\right)}{\sqrt{\pi}}.$$

S3: an approximate solution

- $\boxed{T_{\mathrm{r}}(y) \approx T_{\mathrm{nr}}(y)}$ not significant depend on λ
- $\boxed{\Psi = r^{\nu} T_{\mathrm{nr}}(\theta)}$
- $\boxed{T_{\mathrm{nr}}(\theta) = C_2 y \; {}_2F_1\left(1 - \frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}, 2, y\right)}$
- $\boxed{\Phi = -2\Omega\Psi}$ $\quad \beta = 2$
- error

$\theta \ll 1 \Longrightarrow \theta^2$

$\theta \to \pi/2 \Longrightarrow \cos\theta$

$$\boxed{\frac{1}{T_{\mathrm{r}}}\frac{dT_{\mathrm{r}}}{dy} = \frac{\nu}{s+\nu}\frac{G_1(y) + A_1 G_2(y)}{A_1 G_3(y) + G_4(y)} = \frac{1}{T_{\mathrm{nr}}}\frac{dT_{\mathrm{nr}}}{dy}}$$

$$\Downarrow$$

$$A_1 = D\frac{D_1 + D_2}{D_3 + D_4}$$

$$D = [\beta(s+\nu) - \nu]\frac{\Gamma(1+\beta+\beta s/\nu)\,\Gamma(-\nu/2 - \beta s/2\nu - (\beta+s+1)/2)\,\Gamma(\nu/2 - \beta s/2\nu + (2-\beta+s)/2)}{\Gamma(-\nu/2 + (1+\beta-s)/2 + \beta s/2\nu)\,\Gamma(\nu/2 + (2+\beta+s)/2 + \beta s/2\nu)},$$

$$D_1 = (-2 - \nu + \nu^2)\frac{\Gamma(3/2 - \nu/2)\,\Gamma(1+\nu/2)}{\Gamma(2-\nu/2)\,\Gamma(3/2+\nu/2)},$$

$$D_2 = (\beta - \nu)[\beta(s+\nu) + \nu(1+s+\nu)]\frac{\Gamma(2+\beta+\beta s/\nu)\,\Gamma(-\nu/2 + (1+\beta-s)/2 + \beta s/\nu)\,\Gamma(\nu/2 + (2+\beta+s)/2 + \beta s/\nu)}{\Gamma(-\nu/2 + (2+\beta-s)/2 + \beta s/2\nu)\,\Gamma(\nu/2 + (3+\beta+s)/2 + \beta s/2\nu)},$$

$$D_3 = (2 + \nu - \nu^2)(\beta s + \beta\nu - \nu)\frac{\Gamma(3/2 - \nu/2)\,\Gamma(1+\nu/2)\,\Gamma(1-\beta-\beta s/\nu)}{\Gamma(2-\nu/2)\,\Gamma(3/2+\nu/2)},$$

$$D_4 = (\beta + \nu)[\beta(s+\nu) - \nu(1+s+\nu)]\frac{\Gamma(2-\beta-\beta s/\nu)\,\Gamma(-\nu/2 - (\beta+s-1)/2 - \beta s/2\nu)\,\Gamma(\nu/2 + (2-\beta+s)/2 - \beta s/2\nu)}{\Gamma(-\nu/2 - (\beta+s-2)/2 - \beta s/2\nu)\,\Gamma(\nu/2 + (3-b+s)/2 - \beta s/2\nu)}$$

S3: an approximate solution

error

$\theta \ll 1 \Longrightarrow \theta^2$

$\theta \to \pi/2 \Longrightarrow \cos\theta$

$$\boxed{\begin{aligned}&\Psi = r^{\nu} T_{\rm nr}(\theta) \qquad\qquad 0 \le \nu \le 2\\ &T_{\rm nr}(\theta) = C_2 y\ {}_2F_1\left(1-\frac{\nu}{2}, \frac{1}{2}+\frac{\nu}{2}, 2, y\right)\end{aligned}}$$

- apply: $\theta \ll 1 \text{ or } \theta \to \pi/2$

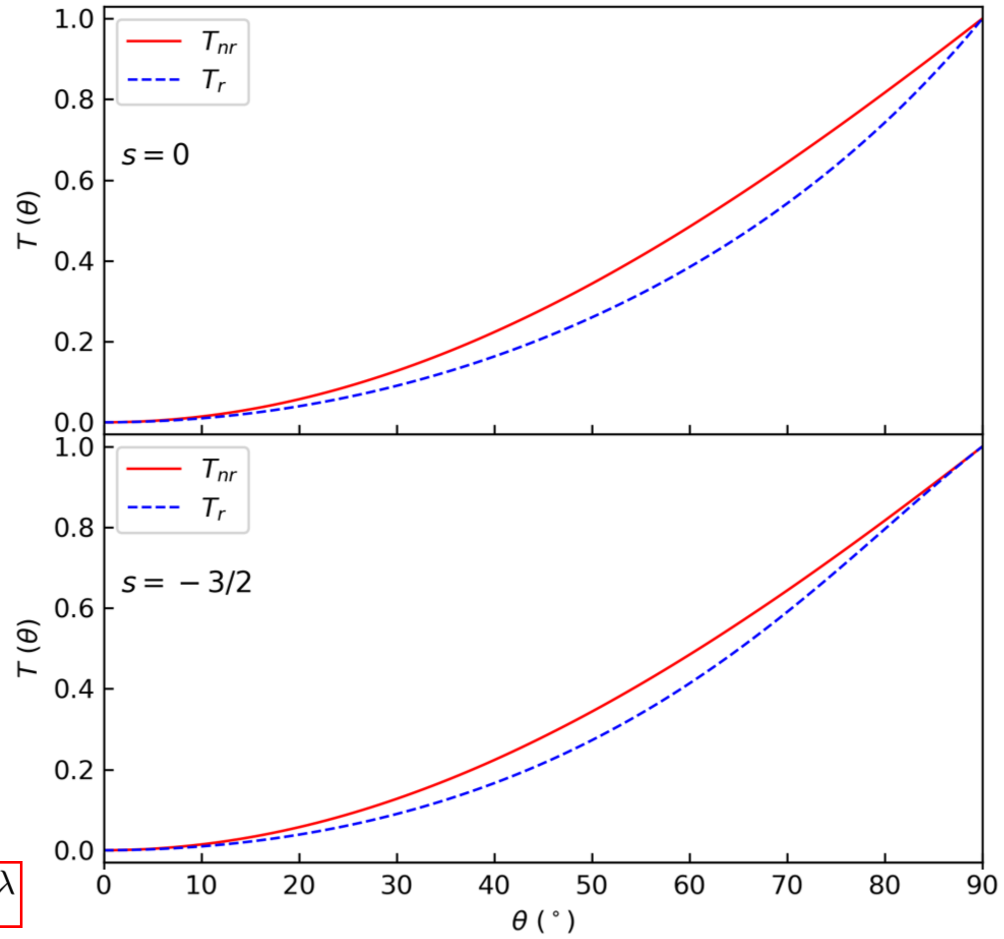
  $\Omega r \sin\theta \gg 1 \text{ or } \Omega r \sin\theta \ll 1$

- independent on angular velocity

$$\Omega = \alpha\Psi^{\lambda} = \alpha R_0^{\lambda\nu} \ (\theta = \pi/2)$$

- self-similar approach: a natural solution!

power law dependence on radius: $\left\{\begin{array}{l}\boxed{\Psi \propto R^{\nu}}\\ \boxed{\Omega \propto R^{\nu\lambda}}\end{array}\right. \Leftarrow \boxed{\Omega = \alpha\Psi^{\lambda}}$



Content

Chen & Zhang 2021, ApJ, 906, 105

S4: magnetic field configuration

- magnetic stream surface (magnetic field line rotating surface)

$$\boxed{\Psi \;=\; r^{\nu} T_{\rm nr}(\theta) = \text{const.}}$$

$$\theta = C_2^{-1/2}\Psi^{1/2} r^{-\nu/2},$$

$$\boxed{R = C_2^{-1/2}\Psi^{1/2} z^{1-\nu/2}}$$

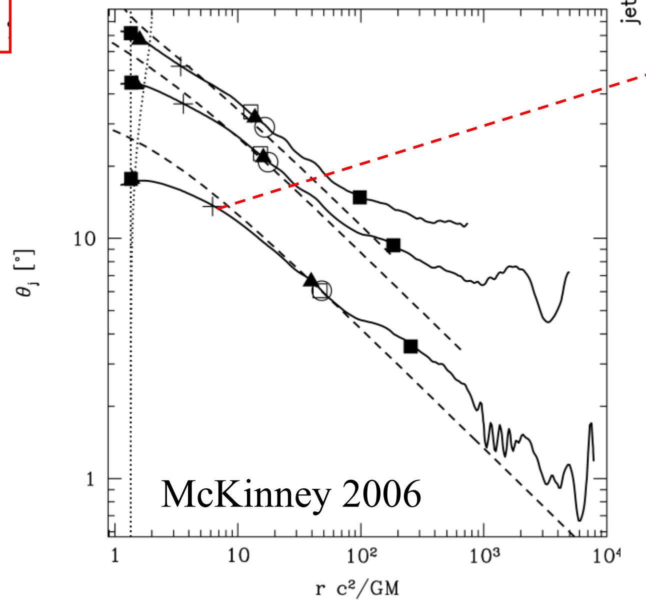
$$\theta \ll 1$$

"general paraboloid"

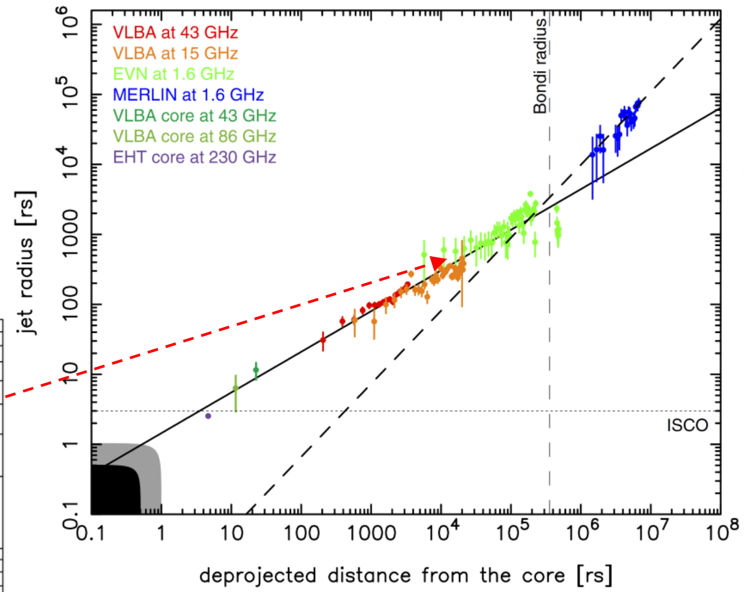
$$R \propto z^{0.56}$$

$$\Downarrow$$

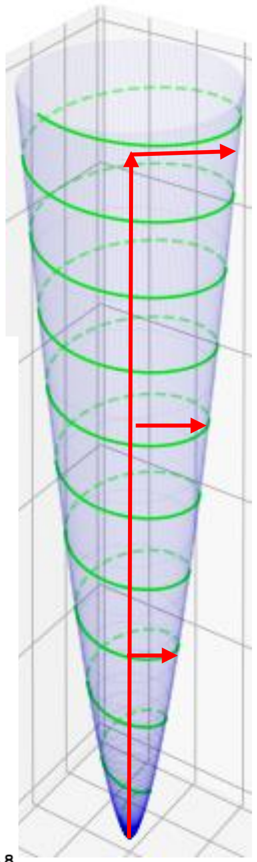
$$\nu = 0.88$$



McKinney 2006



Asada & Nakamura 2012
Nakamura & Asada 2013



S4: magnetic field configuration

- helical magnetic field

$$\frac{-B_\phi}{B_p} = \frac{-\Phi}{|\nabla\Psi|} = \frac{2\Omega\Psi}{|\nabla\Psi|} \simeq \Omega R.$$

$$B_p = \frac{2\Psi}{R^2}$$

$$B_\phi = -\frac{2\Omega\Psi}{R}$$

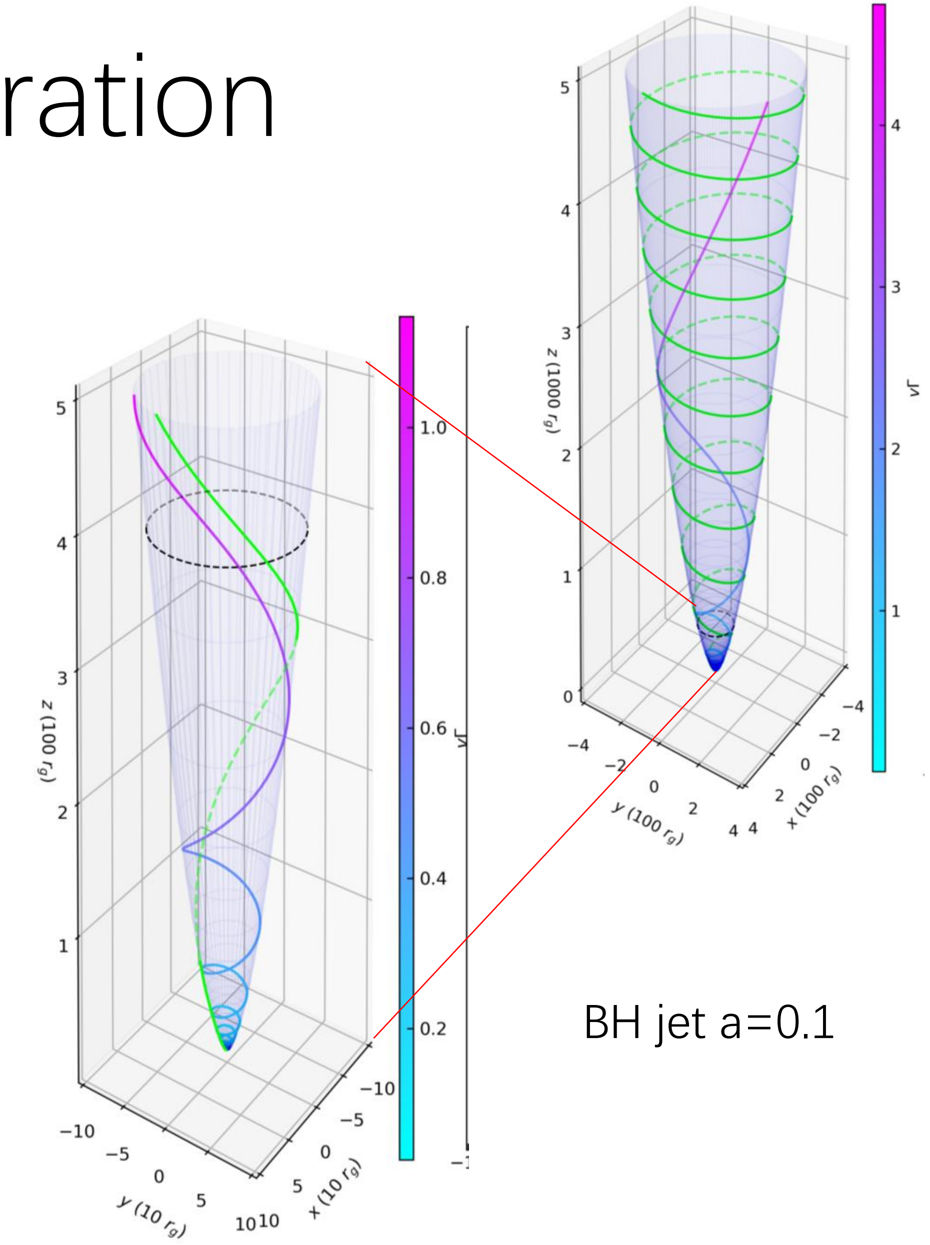
- Alfvén critical surface (light cylinder)

$$\Omega R = 1$$

$$N = \frac{|\phi_{\mathrm{ACS}} - \phi_0|}{2\pi} \approx \frac{C_2^{1/(2-\nu)}}{2\pi\,(\Omega r_0)^{\nu/(2-\nu)}}$$

- magnetic field pitch angle

$$\tan\theta_{\mathrm{inc},\theta\ll 1} = C_2^{1/2}\Omega^{-1}\Psi^{-1/2}r^{-1+\nu/2}$$



BH jet a=0.1

S4: magnetic field configuration

- helical magnetic field

$$\frac{-B_\phi}{B_p} = \frac{-\Phi}{|\nabla\Psi|} = \frac{2\Omega\Psi}{|\nabla\Psi|} \simeq \Omega R.$$

$$B_p = \frac{2\Psi}{R^2}$$
$$B_\phi = -\frac{2\Omega\Psi}{R}$$

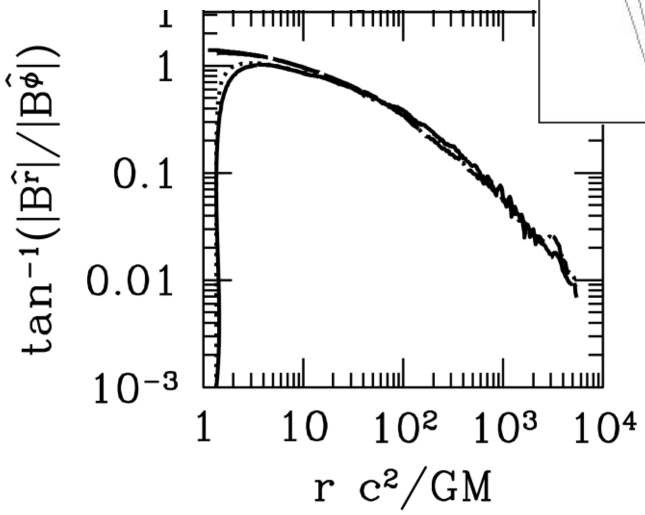
- Alfvén critical surface (light cylinder)

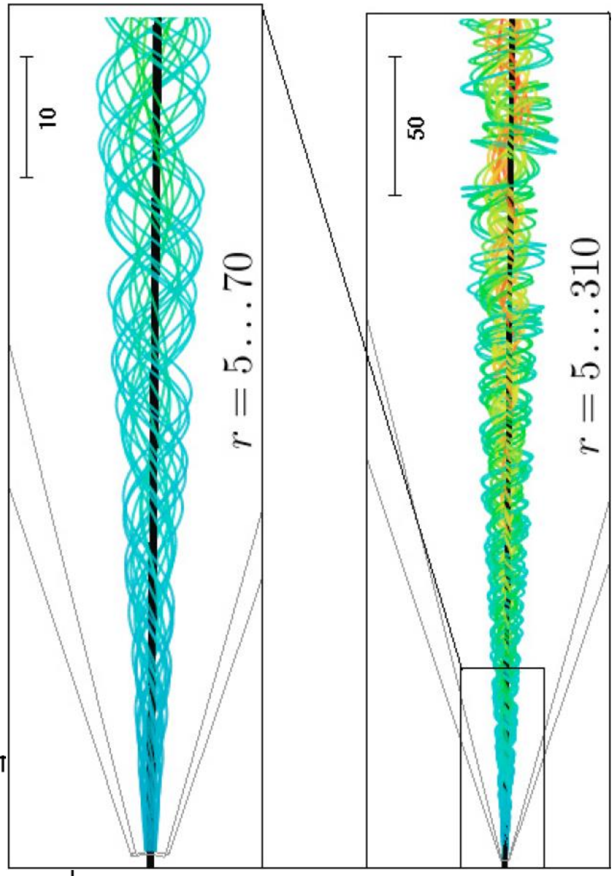
$$\Omega R = 1$$

$$N = \frac{|\phi_{\rm ACS} - \phi_0|}{2\pi} \approx \frac{C_2^{1/(2-\nu)}}{2\pi\,(\Omega r_0)^{\nu/(2-\nu)}}$$

- magnetic field pitch angle

$$\tan\theta_{{\rm inc},\theta\ll 1} = C_2^{1/2}\Omega^{-1}\Psi^{-1/2}r^{-1+\nu/2}$$

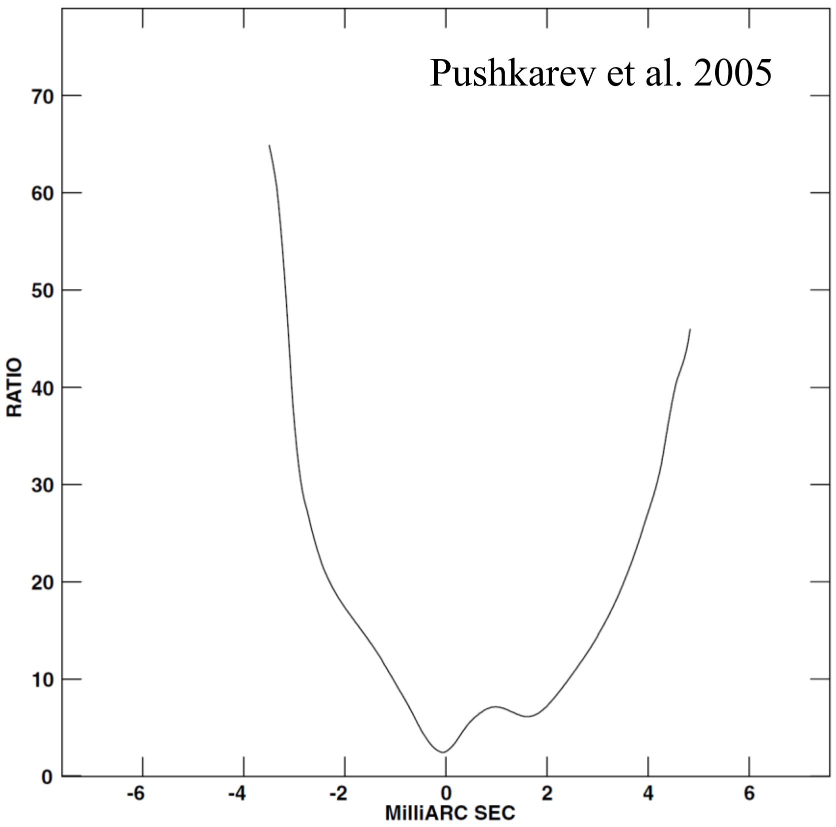




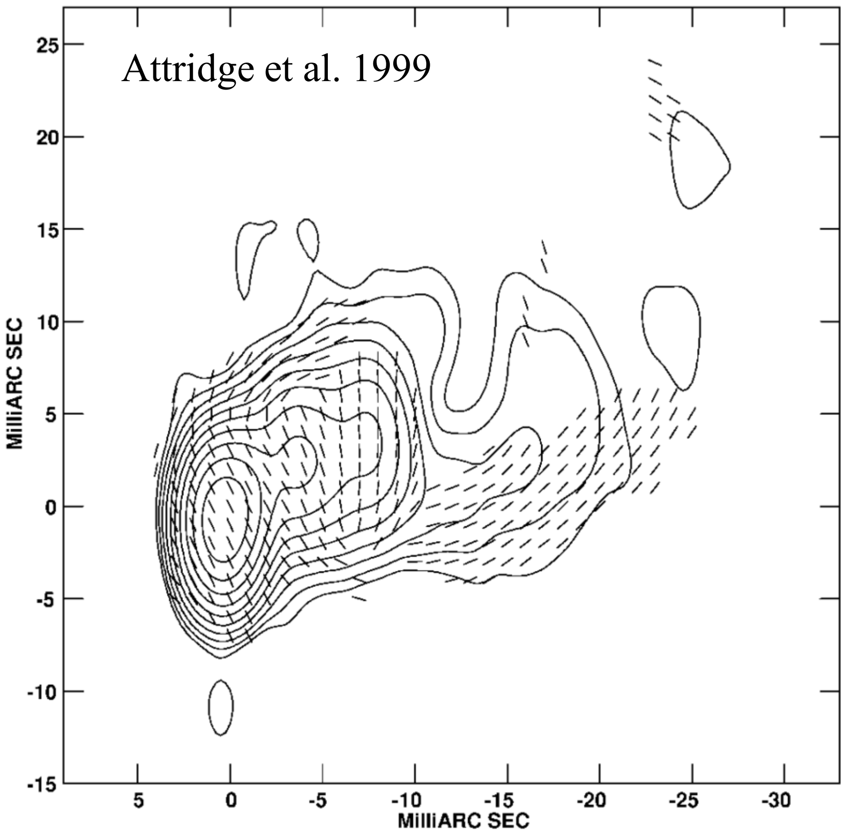
Moll 2009

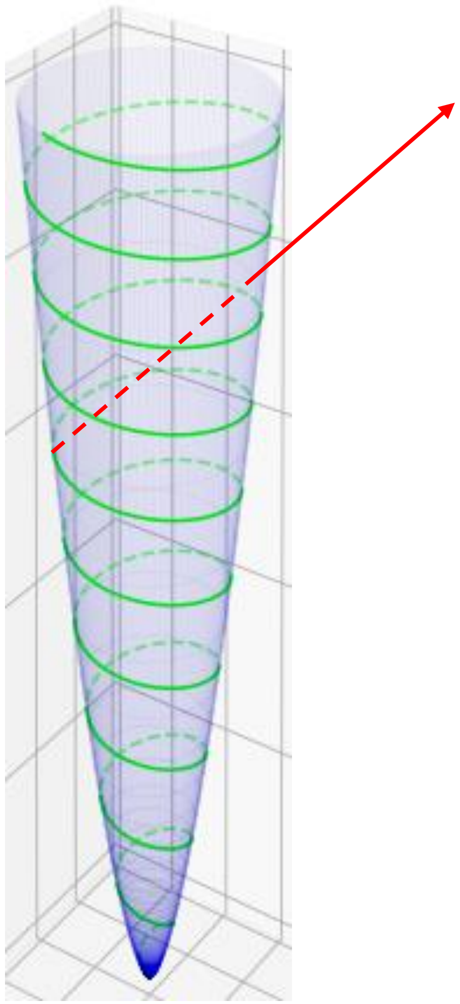
S4: magnetic field configuration – polarization

degree cross jet



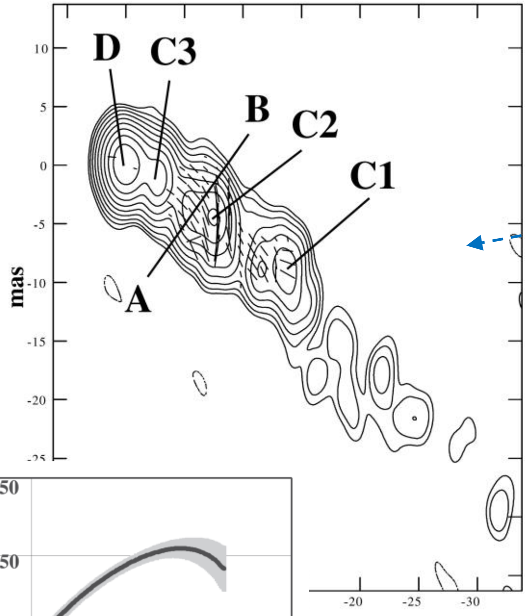
angle cross jet



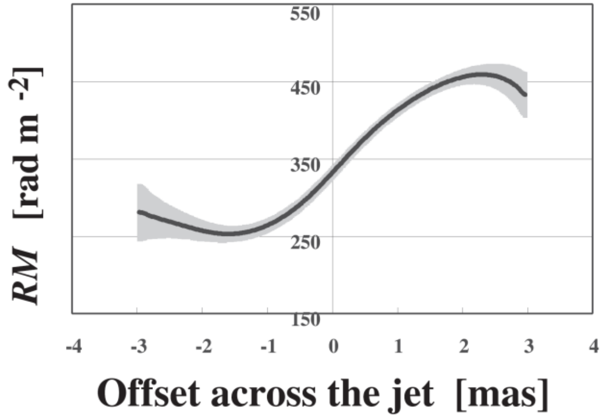


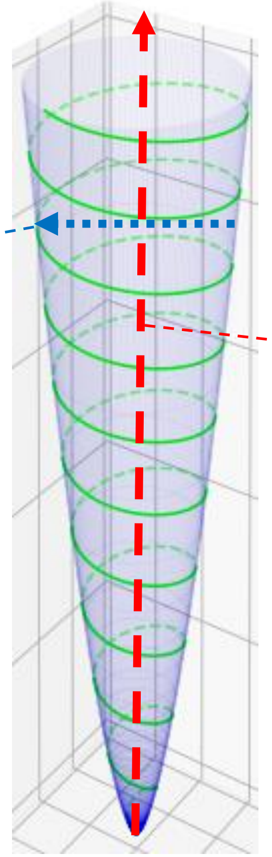
S4: magnetic field configuration –

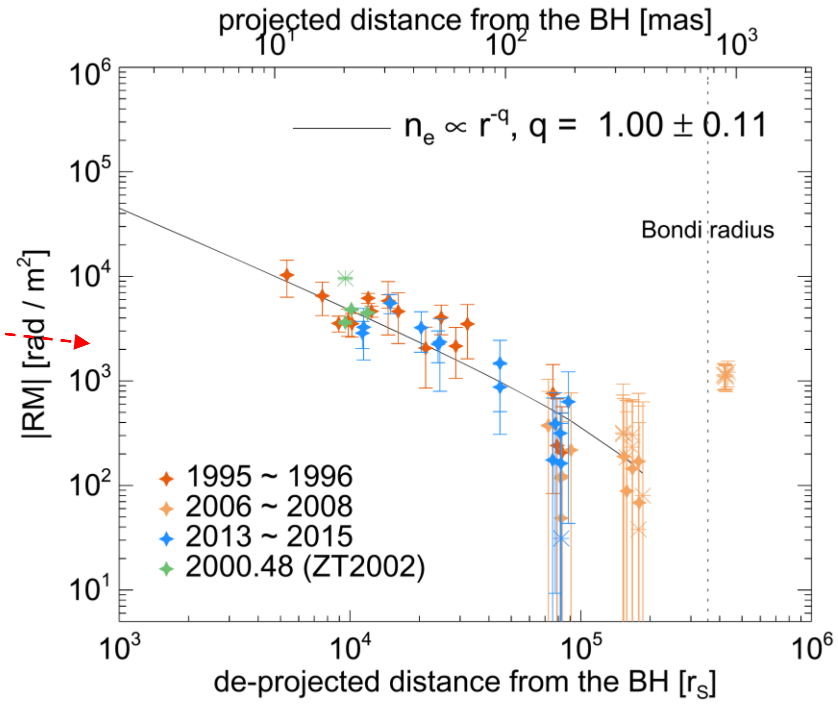
$$\Delta\chi = RM\lambda_{\mathrm{w}}^2$$
$$RM \propto \int n_{\mathrm{e}} B_{\parallel} dl$$



Asada et al. 2002







Park et al. 2019

S4: magnetic field on AD (BP mechanism)

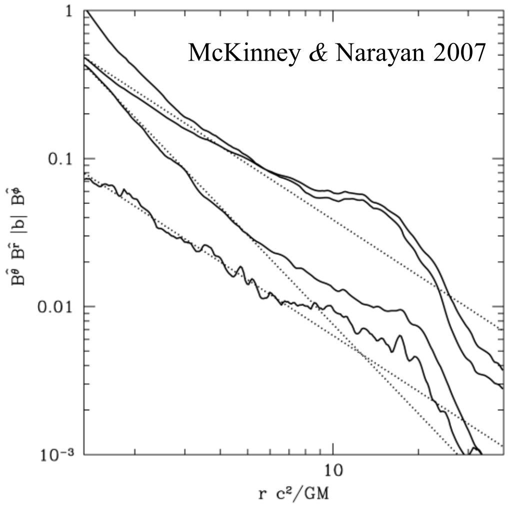
- Strength on AD

$$
\begin{aligned}
B_r &= \frac{r^{\nu-2}}{\sin\theta}\frac{\partial T}{\partial\theta} = \left[C_1 + \nu\left(\nu-1\right)\cos\theta\right] r^{\nu-2}, \\
B_\theta &= -\frac{\nu r^{\nu-2}}{\sin\theta}T = -\nu\left(1 - C_1\cos\theta\right) r^{\nu-2}, \\
B_\phi &= \frac{\Phi}{r\sin\theta} = -\left(1 - C_1\cos\theta\right) 2\Omega r r^{\nu-2}.
\end{aligned}
$$

- Angle with AD mid-plane ($\nu = 3/4$, <60 degree, BP)

$$
\theta_{\mathrm{B,AD}}^{\mathrm{AD}} = \tan^{-1}\left(\left|B_\theta\right|/B_r\right) = \tan^{-1}\left(\nu/C_1\right) \approx \nu\pi/4 = 35^\circ
$$

S4: magnetic field at $\theta \ll 1$



$$\boxed{B_r = 2C_2 r^{\nu-2}} = 2C_2^{2/\nu}\Psi^{1-2/\nu}\theta^{-2+4/\nu},$$

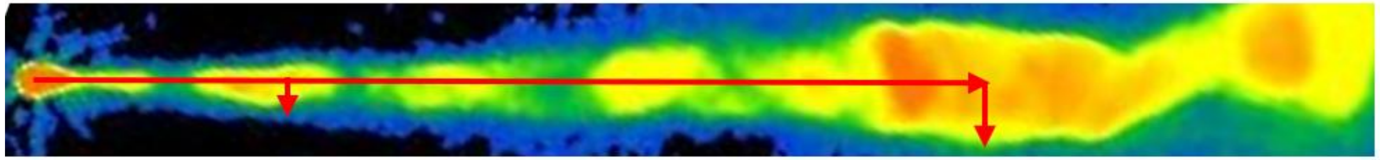
$$\boxed{\begin{aligned} B_\theta &= -\nu C_2^{1/2}\Psi^{1/2}r^{\nu/2-2} = \\ B_\phi &= -2C_2^{1/2}\Omega\Psi^{1/2}r^{-1+\nu/2} \end{aligned}}$$

$$B_\theta = -\nu C_2^{1/2}\Psi^{1/2}r^{\nu/2-2} = -\nu C_2^{2/\nu}\Psi^{1-2/\nu}\theta^{-1+4/\nu} = -\nu C_2 z^{\nu-3} R,$$

$$B_\phi = -2C_2^{1/2}\Omega\Psi^{1/2}r^{-1+\nu/2} = -2C_2^{1/\nu}\Omega\Psi^{1-1/\nu}\theta^{-1+2/\nu} = -2\alpha C_2^{\lambda+1} z^{(\lambda+1)(\nu-2)} R^{2\lambda+1},$$

$$B_R = \frac{2-\nu}{2}\theta B_r,$$

$$B_p = B_z = B_r.$$



future observations

S5: jet velocity & acceleration

- general velocity

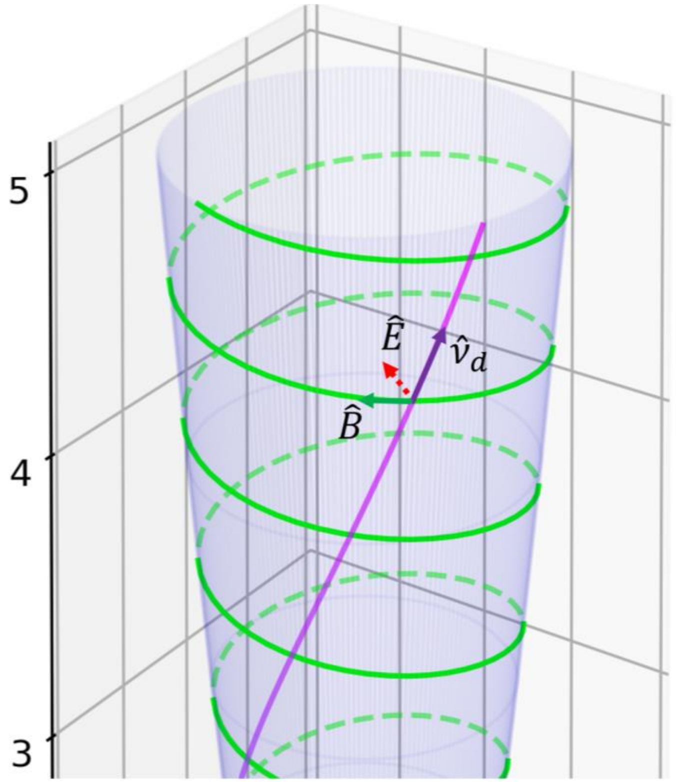
$$\mathbf{v} = \Omega r \sin\theta \hat{\phi} + \kappa \mathbf{B} = \mathbf{E} \times \mathbf{B}/B^2 + \zeta \mathbf{B}$$

- force-free: ignore inertia
- Poynting flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}/4\pi = \mathbf{v}_{\mathrm{d}} B^2/4\pi$$

- drift velocity (S7)

$$\mathbf{v} = \mathbf{v}_{\mathrm{d}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \Omega r \sin\theta \left(\hat{\phi} - \frac{B_\phi}{B^2} \mathbf{B} \right) = \Omega r \sin\theta \left(\frac{B_p^2}{B^2} \hat{\phi} - \frac{B_\phi}{B^2} \mathbf{B}_p \right)$$



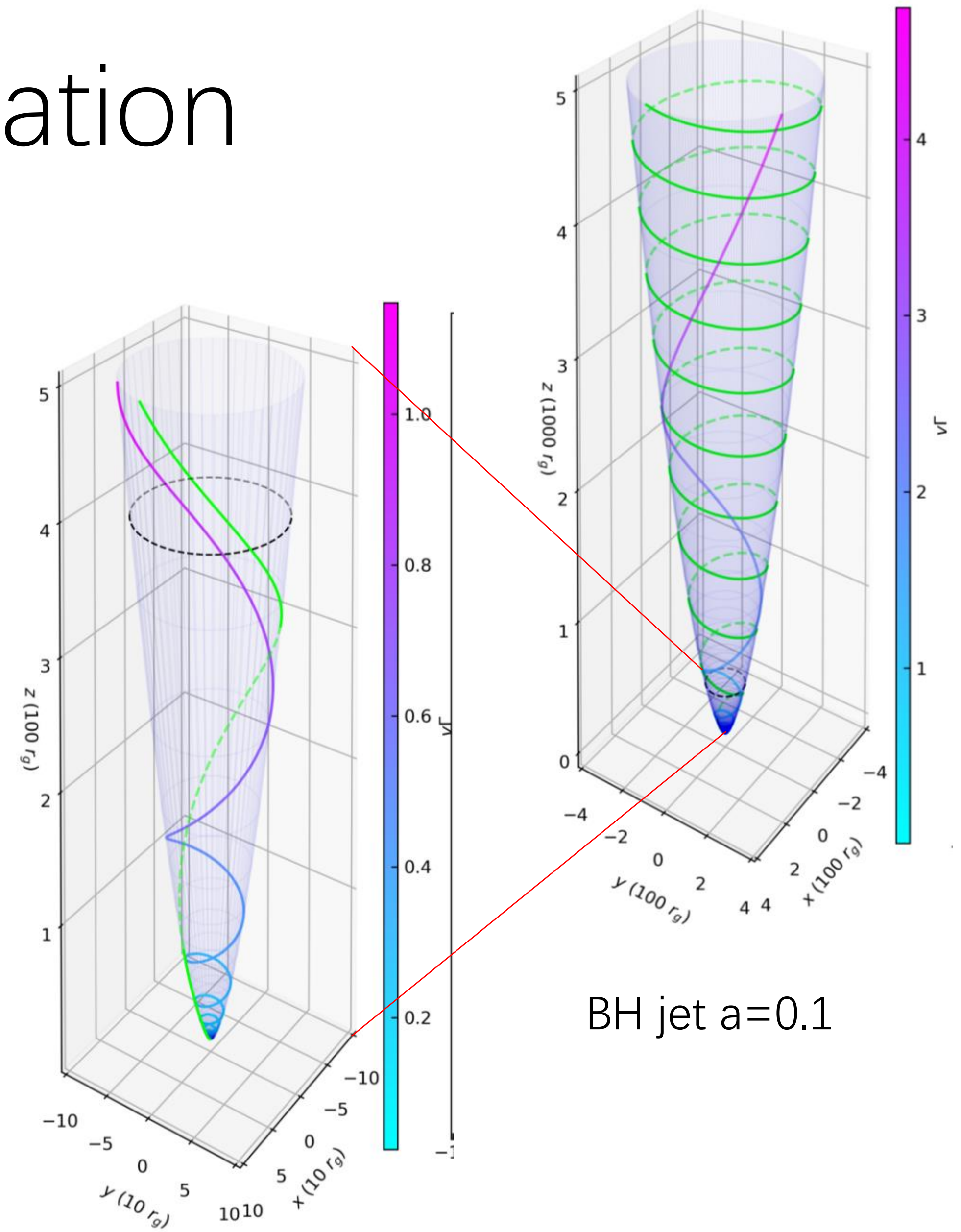
S5: jet velocity & acceleration

- perpendicular to magnetic field

$$v_p / v_\phi = -B_\phi / B_p$$

- individual components

$$\begin{aligned} v_\phi &= \Omega r \sin\theta \frac{B_p^2}{B^2} \approx \frac{\Omega R}{1+(\Omega R)^2}, \\ v_p &= -\Omega r \sin\theta \frac{B_\phi B_p}{B^2} \approx \frac{(\Omega R)^2}{1+(\Omega R)^2}, \\ v &= \Omega r \sin\theta \frac{B_p}{B} \approx \frac{\Omega R}{\sqrt{1+(\Omega R)^2}}, \\ \boxed{v\Gamma} &\boxed{\approx \quad \Omega R.} \end{aligned}$$



BH jet a=0.1

S5: jet velocity & acceleration

- causality constrain

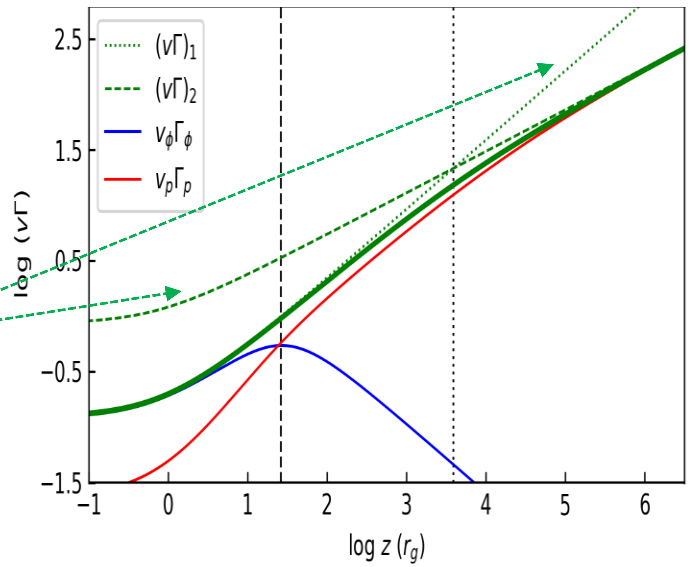
$$\frac{1}{(v\Gamma)^2} = \frac{1}{(v\Gamma)_1^2} + \frac{1}{(v\Gamma)_2^2},$$

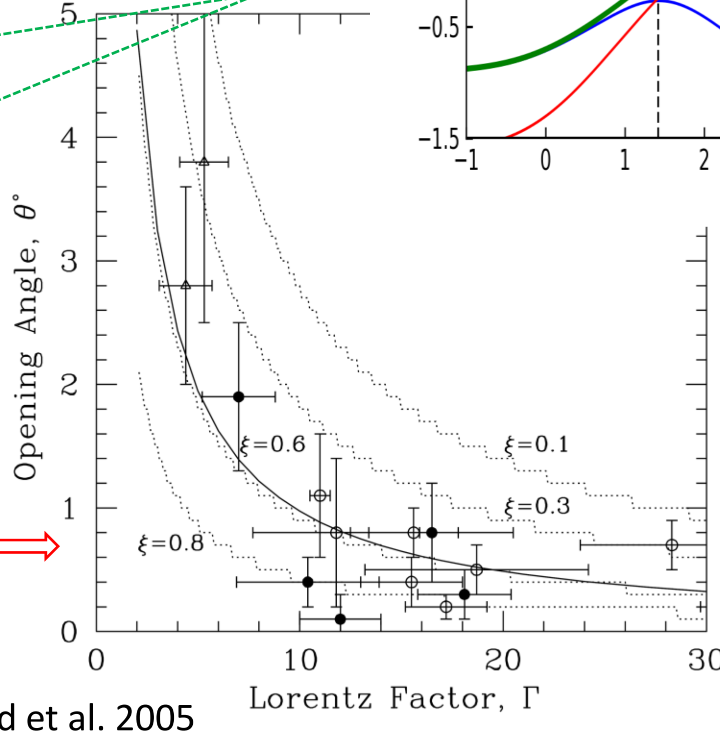
$$\boxed{(v\Gamma)_{1,\theta\ll 1} \;=\; \Omega R}$$

$$\boxed{(v\Gamma)_{2,\theta\ll 1} \;=\; \frac{2}{\sqrt{2-\nu}}\theta^{-1}}$$

- jet velocity versus opening angle

$$\boxed{(v\Gamma) \;\lesssim\; 1/\theta}$$





Jorstad et al. 2005

S5: jet velocity & acceleration

- toroidal velocity

$$\boxed{\frac{1}{v_\phi} = \frac{1}{\Omega r \sin\theta} + \frac{1}{\Omega r \sin\theta}\frac{B_\phi^2}{B_p^2} = \frac{1}{(v_\phi)_1} + \frac{1}{(v_\phi)_2}}$$

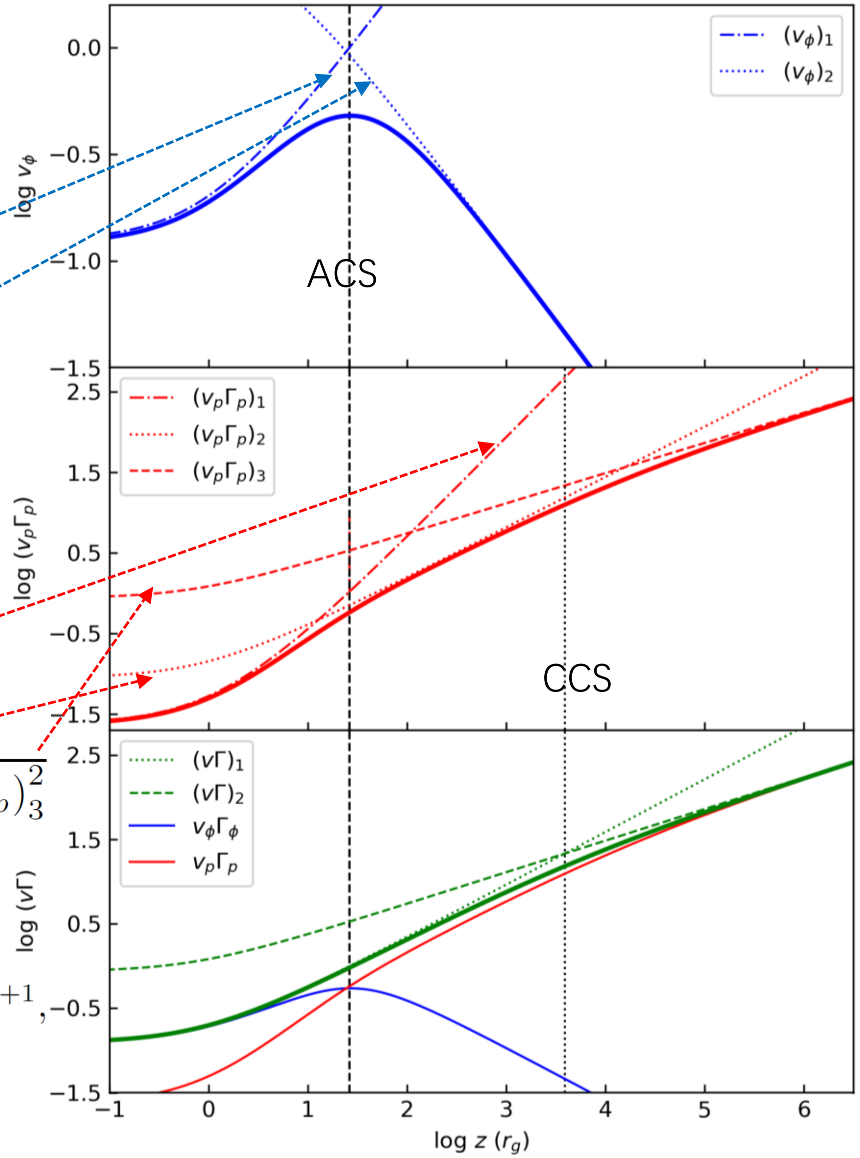
- poloidal velocity

$$\frac{1}{(v_p\Gamma_p)^2} = \frac{B_p^2}{(\Omega r \sin\theta B_\phi)^2} + \frac{2}{(\Omega r \sin\theta)^2} + \frac{B_\phi^2}{(\Omega r \sin\theta B_p)^2} - 1 = \frac{1}{(v_p\Gamma_p)_1^2} + \frac{1}{(v_p\Gamma_p)_2^2} + \frac{1}{(v_p\Gamma_p)_3^2}$$

- three stages

$$V_{1,\theta\ll 1} = C_2^{-1/\nu}\Omega\Psi^{1/\nu}\theta^{-(2-\nu)/\nu} = C_2^{-1/2}\Omega\Psi^{1/2}r^{1-\nu/2} = \alpha C_2^{\lambda} z^{-\lambda(2-\nu)} R^{2\lambda+1},$$

$$V_{2,\theta\ll 1} = \frac{2}{\sqrt{2-\nu}}\theta^{-1} = \frac{2C_2^{1/2}}{\sqrt{2-\nu}}\Psi^{-1/2}r^{\nu/2} = \frac{2}{\sqrt{2-\nu}} z R^{-1} = \sqrt{\frac{\nu\mathcal{R}}{R}},$$



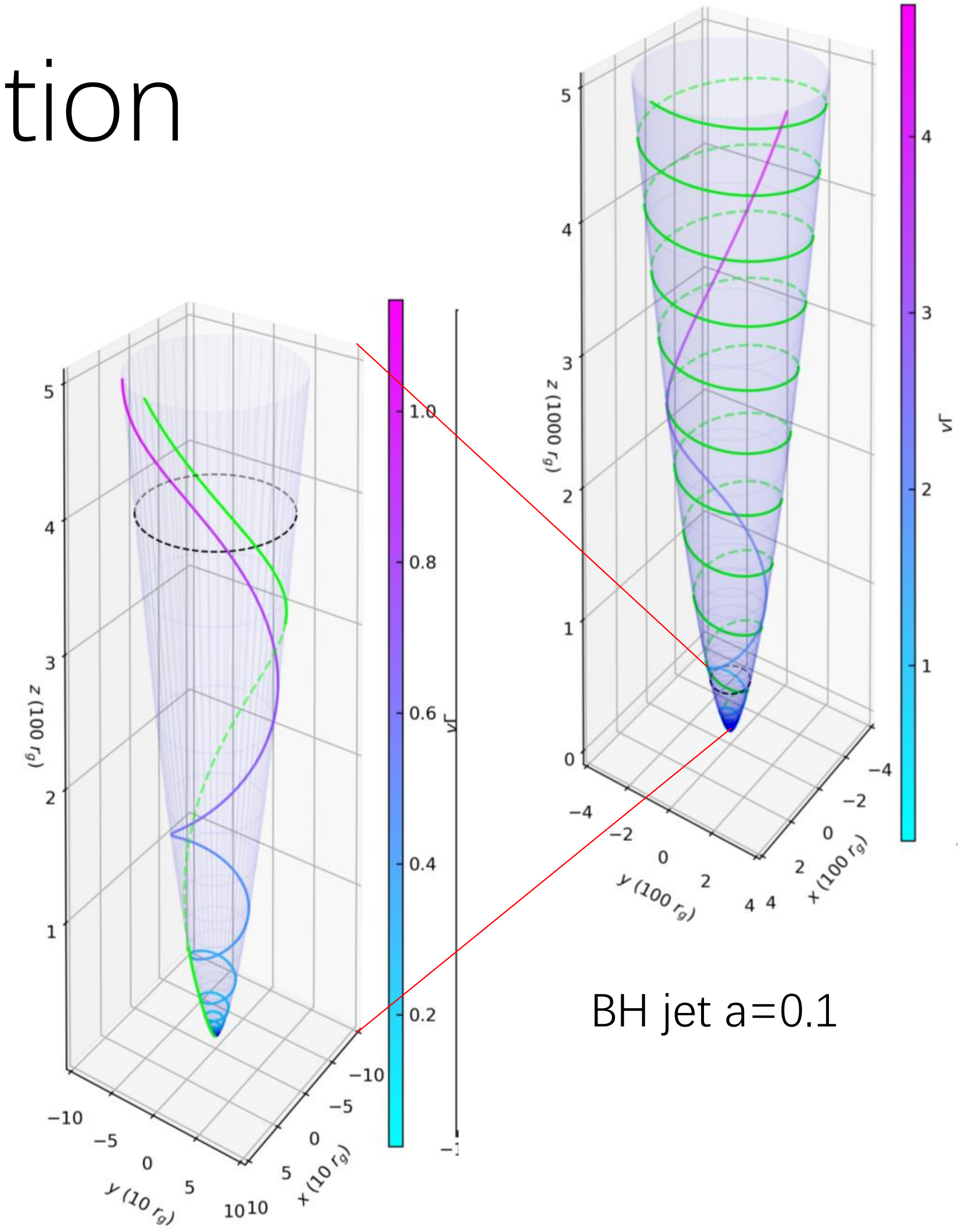
S5: jet velocity & acceleration

three acceleration stages

AS I	ACS	AS II	CCS	AS III

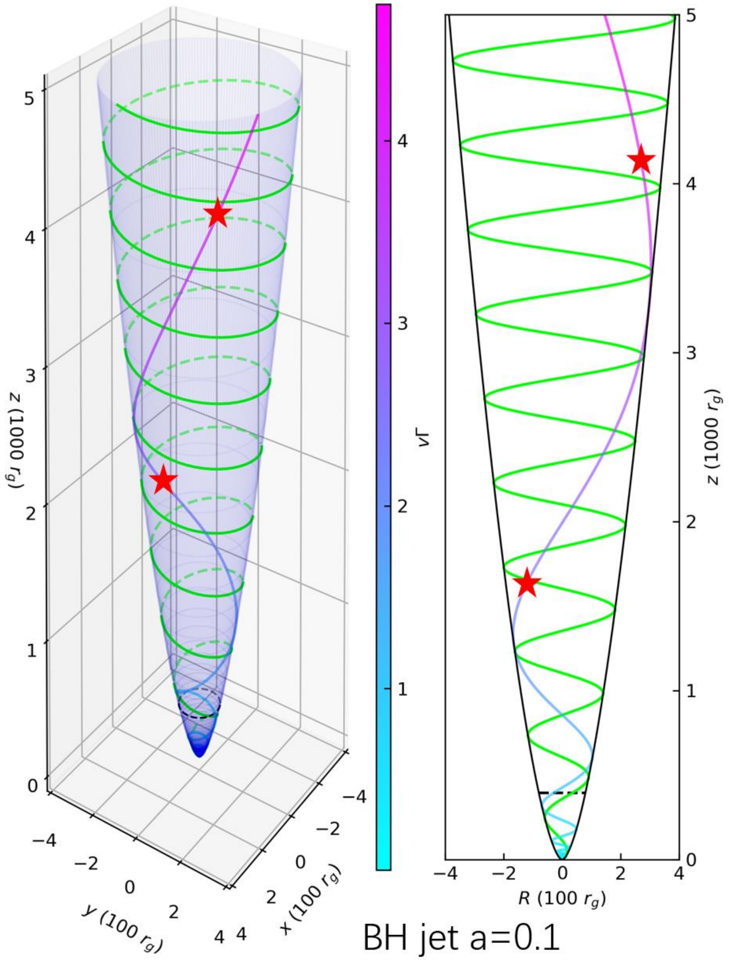
$\Omega R < 1$	$\Omega R > 1$	$\Omega R \gtrsim 2/\left(\theta\sqrt{2-\nu}\right)$
$B_\phi < B_p$	$B_p < B_\phi$	$B_p < B_\phi$
$v\Gamma = \Omega R$	$v\Gamma = \Omega R$	$v\Gamma = 2/\left(\theta\sqrt{2-\nu}\right)$
$v_p\Gamma_p = (\Omega R)^2$	$v_p\Gamma_p = \Omega R/\sqrt{2}$	$v_p\Gamma_p = 2/\left(\theta\sqrt{2-\nu}\right)$
$v_\phi = \Omega R$	$v_\phi = 1/\Omega R$	$v_\phi = 1/\Omega R$

$$v_p/v_\phi = -B_\phi/B_p$$



BH jet a=0.1

S5: jet velocity & acceleration



BH jet a=0.1

jet flow is helical

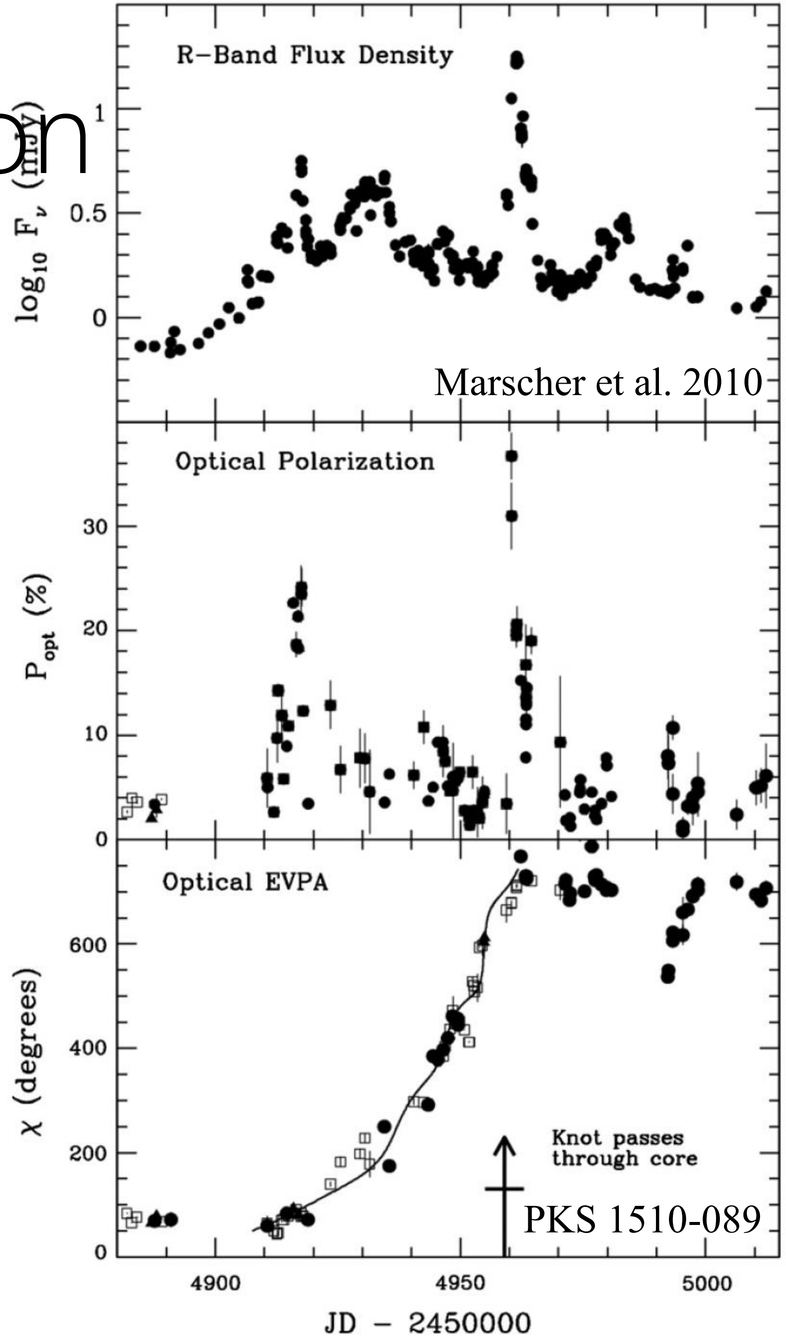
$$v_p/v_\phi = -B_\phi/B_p$$

$$P = \frac{2\pi r\sin\theta}{v_\phi} = \frac{2\pi r\sin\theta}{(v_\phi)_1} + \frac{2\pi r\sin\theta}{(v_\phi)_2} = P_1 + P_2,$$

$$P_1 = \frac{2\pi r\sin\theta}{(v_\phi)_1} = \frac{2\pi}{\Omega},$$

$$P_2 = \frac{2\pi}{\Omega}\frac{B_\phi^2}{B_p^2} \approx 2\pi\Omega\,(r\sin\theta)^2 = \frac{2\pi}{\Omega}V_1^2.$$

polarization angle
continuous changing

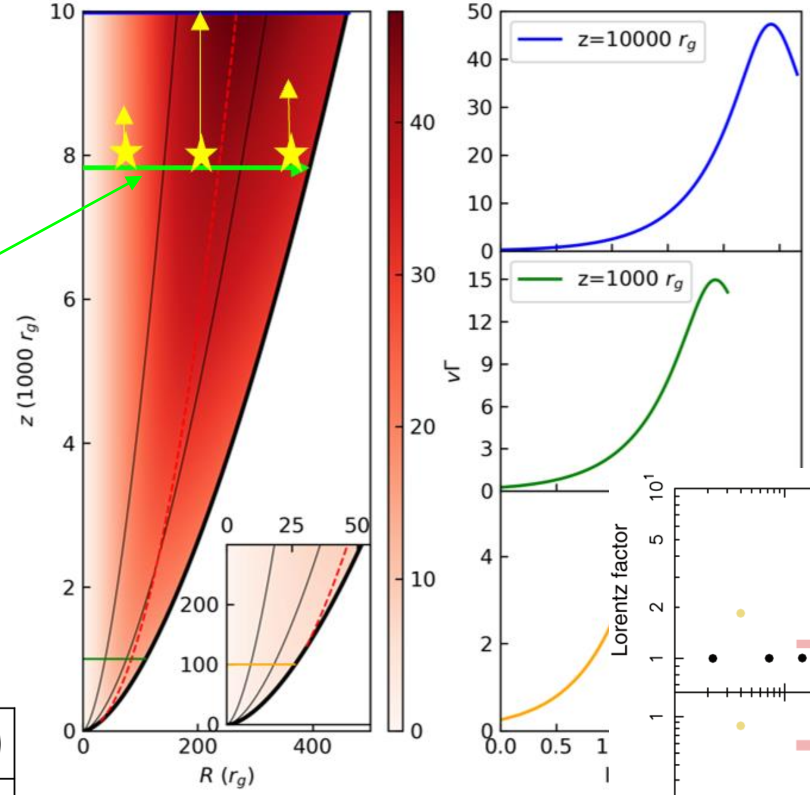


S5: jet velocity & acceleration

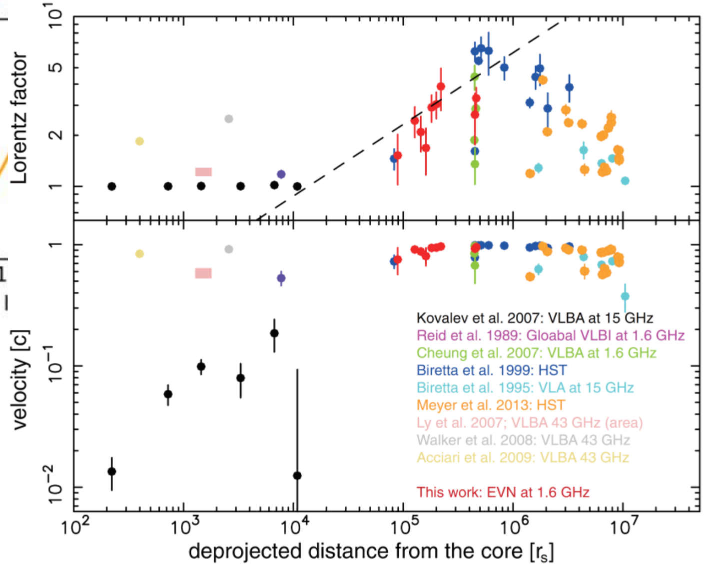
jet flow stratified:

Poynting flux and Lorentz force vanish on axis

$$v\Gamma = \Omega R$$



BH jet
a=1



ACS	CCS	
$\Omega R < 1$	$\Omega R > 1$	$\Omega R \gtrsim 2/\left(\theta\sqrt{2-\nu}\right)$
$B_\phi < B_p$	$B_p < B_\phi$	$B_p < B_\phi$
$v\Gamma = \Omega R$	$v\Gamma = \Omega R$	$v\Gamma = 2/\left(\theta\sqrt{2-\nu}\right)$
$v_p\Gamma_p = (\Omega R)^2$	$v_p\Gamma_p = \Omega R/\sqrt{2}$	$v_p\Gamma_p = 2/\left(\theta\sqrt{2-\nu}\right)$
$v_\phi = \Omega R$	$v_\phi = 1/\Omega R$	$v_\phi = 1/\Omega R$

S6: current & charge

The plasma supplies currents and charges as needed to support the electromagnetic field.

$$\mathbf{j}_p = \frac{\Phi' \mathbf{B}_p}{4\pi} = -\frac{(\lambda+1)\,\Omega \mathbf{B}_p}{2\pi}$$

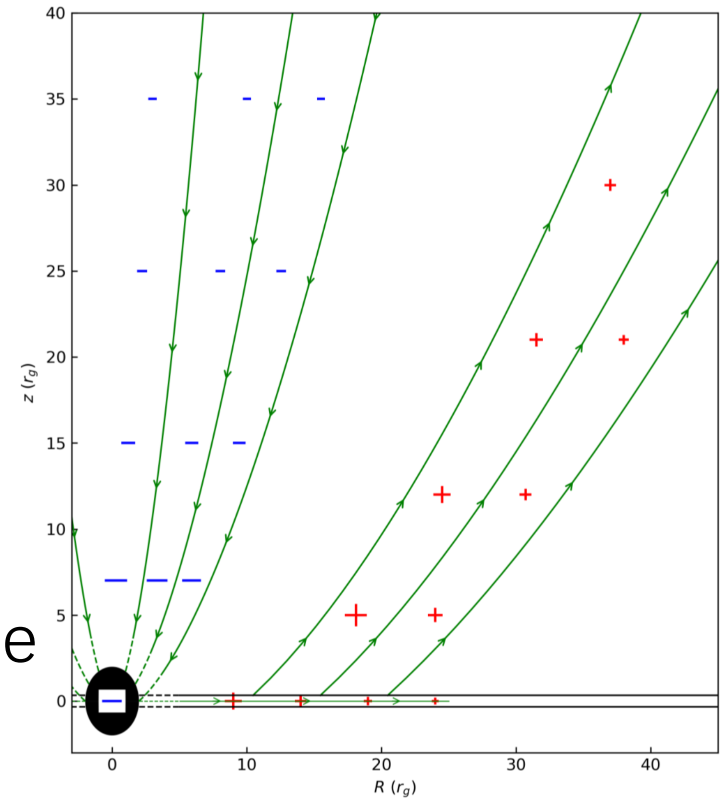
$$j_\phi = \frac{-1}{4\pi R}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial R^2} - \frac{1}{R}\frac{\partial \Psi}{\partial R}\right) = \begin{cases} \approx 0 & \Omega R > 1 \text{ or } \theta \ll 1, \\ \ll j_p & \Omega R < 1 \;\&\; \theta \to \pi/2. \end{cases}$$

- jet flow has to be charged

$$\rho_{\rm e} = \frac{\nabla \cdot \mathbf{E}}{4\pi} = -\frac{\mathbf{\Omega} \cdot \mathbf{B}}{2\pi} + \Omega r \sin\theta j_\phi - \frac{\Omega'}{4\pi}\left|\nabla \Psi\right|^2 \approx j_p$$

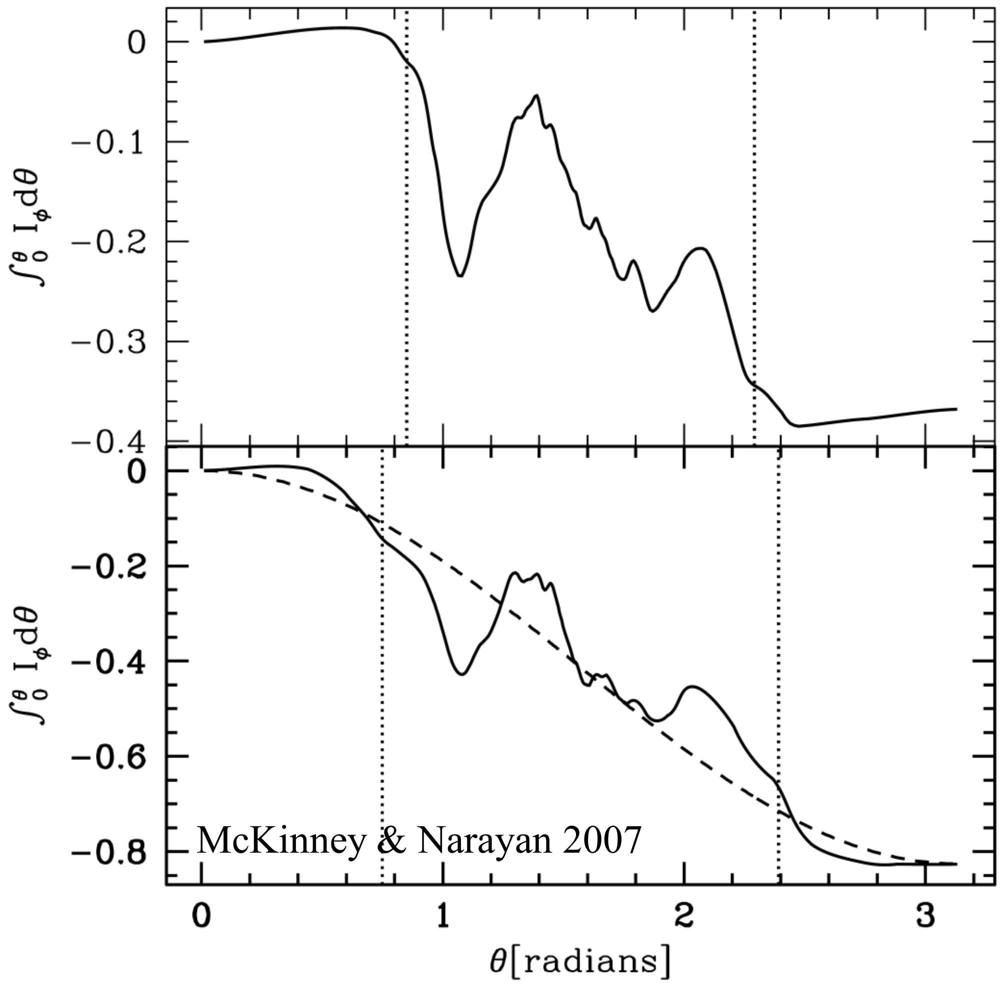
- electric force can be ignored in non-relativistic case

$$\mathbf{F}_{\rm L} = \rho\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \rho_{\rm e}\mathbf{E} + \mathbf{j}\times\mathbf{B} \qquad \rho_{\rm e} \neq 0$$



S6: current & charge

$$j_\phi = \begin{cases} \approx 0 & \Omega R > 1 \text{ or } \theta \ll 1, \\ \ll j_p & \Omega R < 1 \ \& \ \theta \to \pi/2. \end{cases}$$



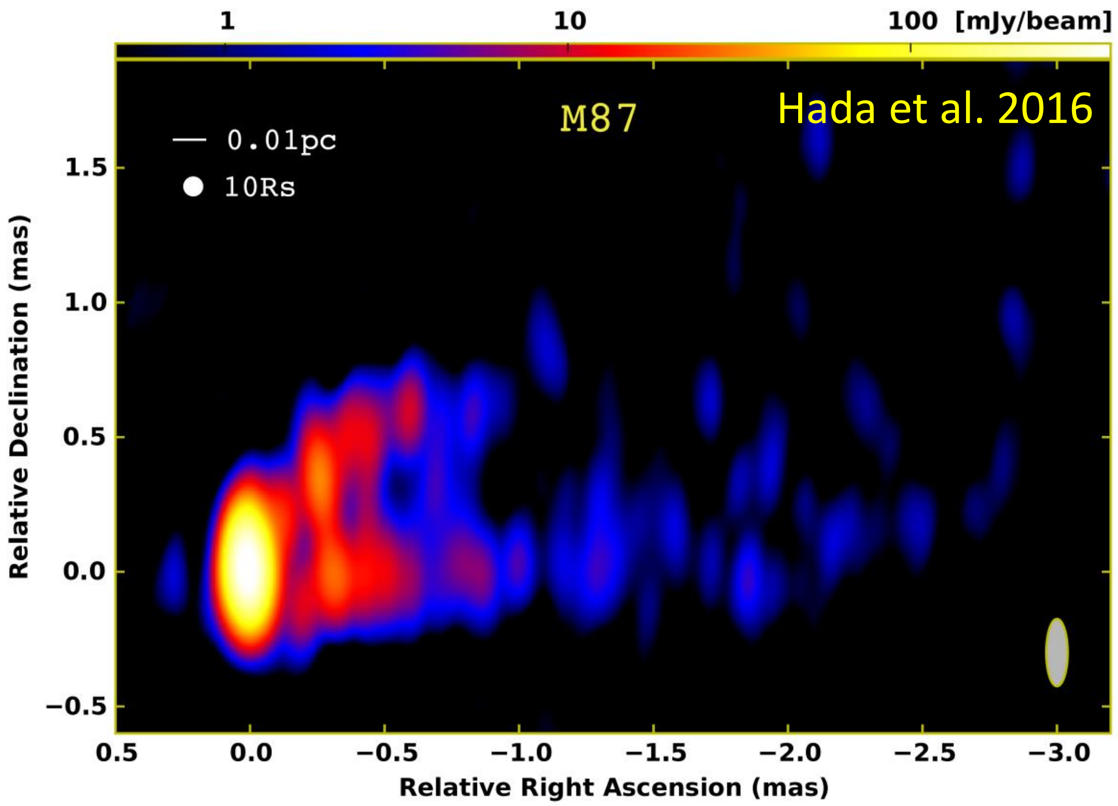
S6: jet power & electric potential difference

- Hollow jet

  no Poynting flux on axis

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}/4\pi$$

$$S_r = -\frac{\Omega R}{4\pi} B_\phi B_r = \frac{\Omega^2}{2\pi} \frac{\Psi}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} = \frac{B_\phi^2}{8\pi} \frac{\sin\theta}{T} \frac{dT}{d\theta},$$
$$S_z = -\frac{\Omega R}{4\pi} B_\phi B_z = -\frac{\Omega \Phi}{4\pi R} \frac{\partial \Psi}{\partial R} = \frac{B_\phi^2}{8\pi} \sin^2\theta \left(\frac{1}{T} \frac{dT}{d\theta} \cot\theta + \nu \right)$$



$$S_z = \frac{B_\phi^2}{4\pi} = \frac{C_2}{\pi} \Omega^2 \Psi r^{\nu-2} = \frac{C_2^{2/\nu}}{\pi} \Omega^2 \Psi^{2-2/\nu} \theta^{-2+4/\nu} = \frac{\alpha^2}{\pi} C_2^{2\lambda+2} z^{2\lambda\nu+2\nu-4\lambda-4} R^{4\lambda+2}$$

S6: jet power & electric potential difference

- Jet power (between two layers)

$$P_{\rm jet} = 2\int_{\theta_1}^{\theta_2} S_r 2\pi r^2 \sin\theta d\theta = \left.\frac{\Omega^2\Psi^2}{|\lambda+1|}\right|_1^2 = \left.\frac{\Omega^2 F_B^2}{4\pi^2|\lambda+1|}\right|_1^2 = \left.\frac{R^2 B_\phi^2}{4|\lambda+1|}\right|_1^2 = \left.\frac{J^2}{|\lambda+1|}\right|_1^2$$

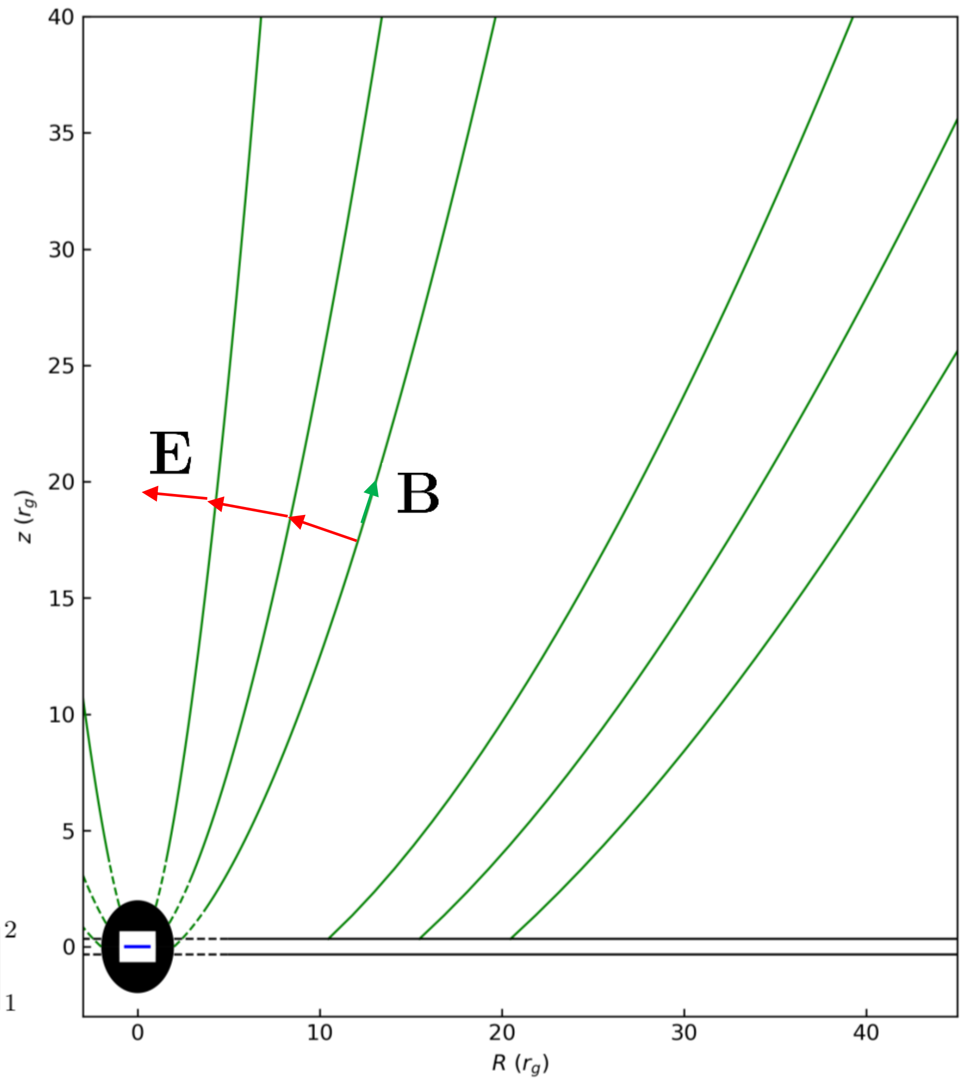
- The CO case $\lambda = 0$

$$\boxed{P_{\rm jet} = J^2 = \Delta V^2}$$

- Potential difference

$$\Delta V = \int_{l_1}^{l_2} \mathbf{E}\cdot d\mathbf{l}_{\rm E}$$

$$= -\int_{l_1}^{l_2} \Omega\nabla\Psi\cdot d\mathbf{l}_{\rm E} = -\left.\frac{\Omega\Psi}{(\lambda+1)}\right|_1^2 = -\left.\frac{\Omega F_B}{2\pi(\lambda+1)}\right|_1^2 = \left.\frac{RB_\phi}{2(\lambda+1)}\right|_1^2 = \left.\frac{J}{(\lambda+1)}\right|_1^2$$



S6: jet power & electric potential difference

- jet current (3C 303)

$$J = \sqrt{P_{\rm jet}} \approx 5.8 \times 10^{17} \sqrt{P_{44}}\ {\rm A}$$

$\sim 3.9 \times 10^{18}\ {\rm A}$ Kronberg et al. 2011

$\sim 1.0 \times 10^{46}\ {\rm erg\ s^{-1}}$ Zhang et al. 2018

- potential difference (Crab nebula)

$$\Delta V = \sqrt{P_{\rm jet}} \approx 1.7 \times 10^{19} \sqrt{P_{44}} \approx 5 \times 10^{15}\ {\rm volts}$$

$\gtrsim 10^{37}\ {\rm erg\ s^{-1}}$ Hester 2008

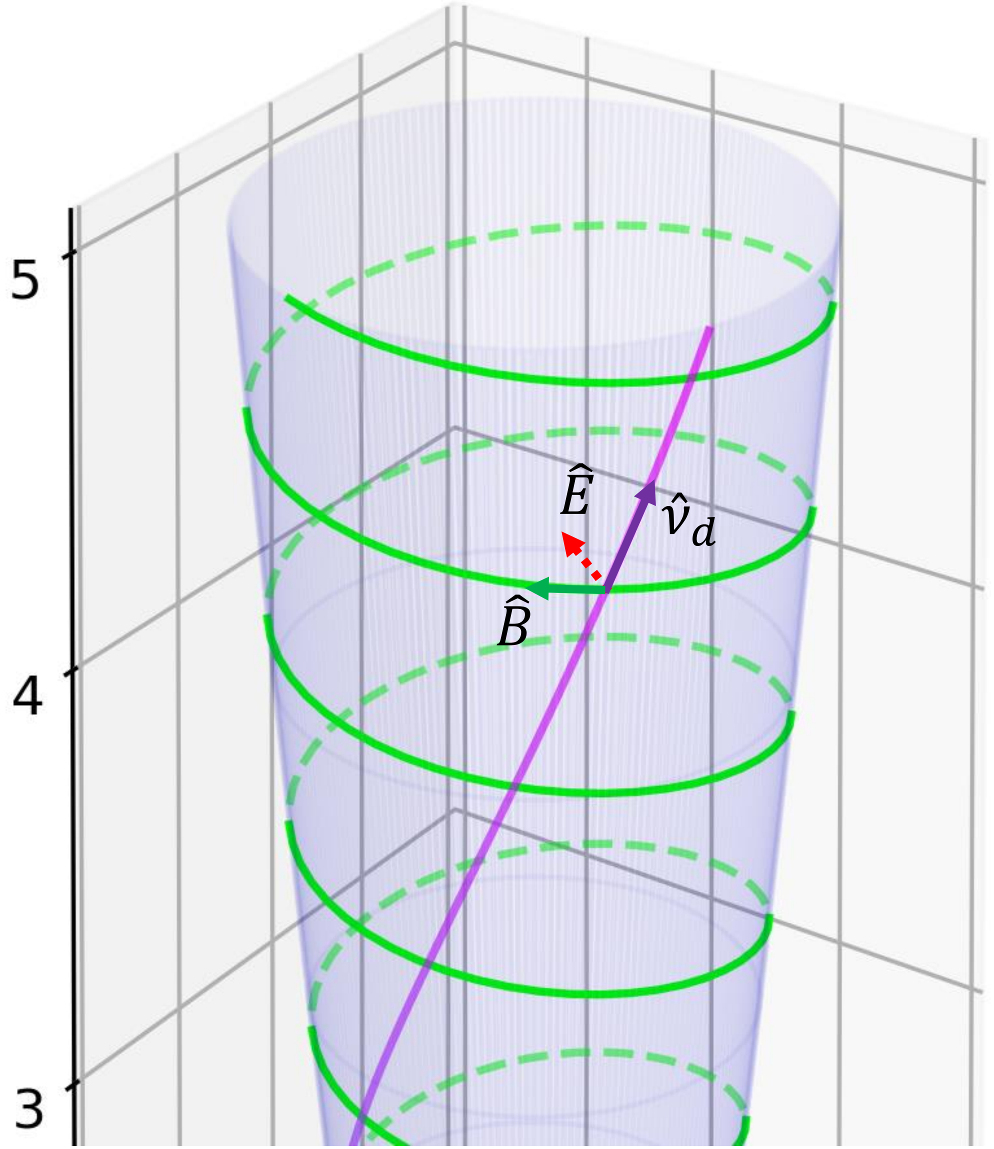
$\approx 0.45\ {\rm PeV}$ Amenomori et al. 2019

S7: jet dynamics

- The force (I)

$$\boxed{\begin{aligned} \mathbf{F}_\mathrm{L} &= \rho\left(\mathbf{u}\cdot\bigtriangledown\right)\mathbf{u} = \rho_\mathrm{e}\mathbf{E} + \mathbf{j}\times\mathbf{B} \\ \mathbf{F}_{\mathrm{L},\hat{\mathrm{B}}} &= 0 \end{aligned}}$$

$$\mathbf{F}_\mathrm{L} = \overbrace{\underbrace{\mathbf{j}_p\times\mathbf{B}_p}_{1,\hat\phi} + \underbrace{\left(\mathbf{j}_p\times\mathbf{B}_\phi\right)_{B_p}\hat{B}_p}_{2,\hat{B}_p}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{v}}_\mathrm{d}}} + \overbrace{\underbrace{\left(\mathbf{j}_p\times\mathbf{B}_\phi\right)_E\hat{E}}_{3,\hat{E}} + \underbrace{\mathbf{j}_\phi\times\mathbf{B}_p}_{4,\hat{E}} + \underbrace{\rho_\mathrm{e}\mathbf{E}}_{5,\hat{E}}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{E}}}}$$



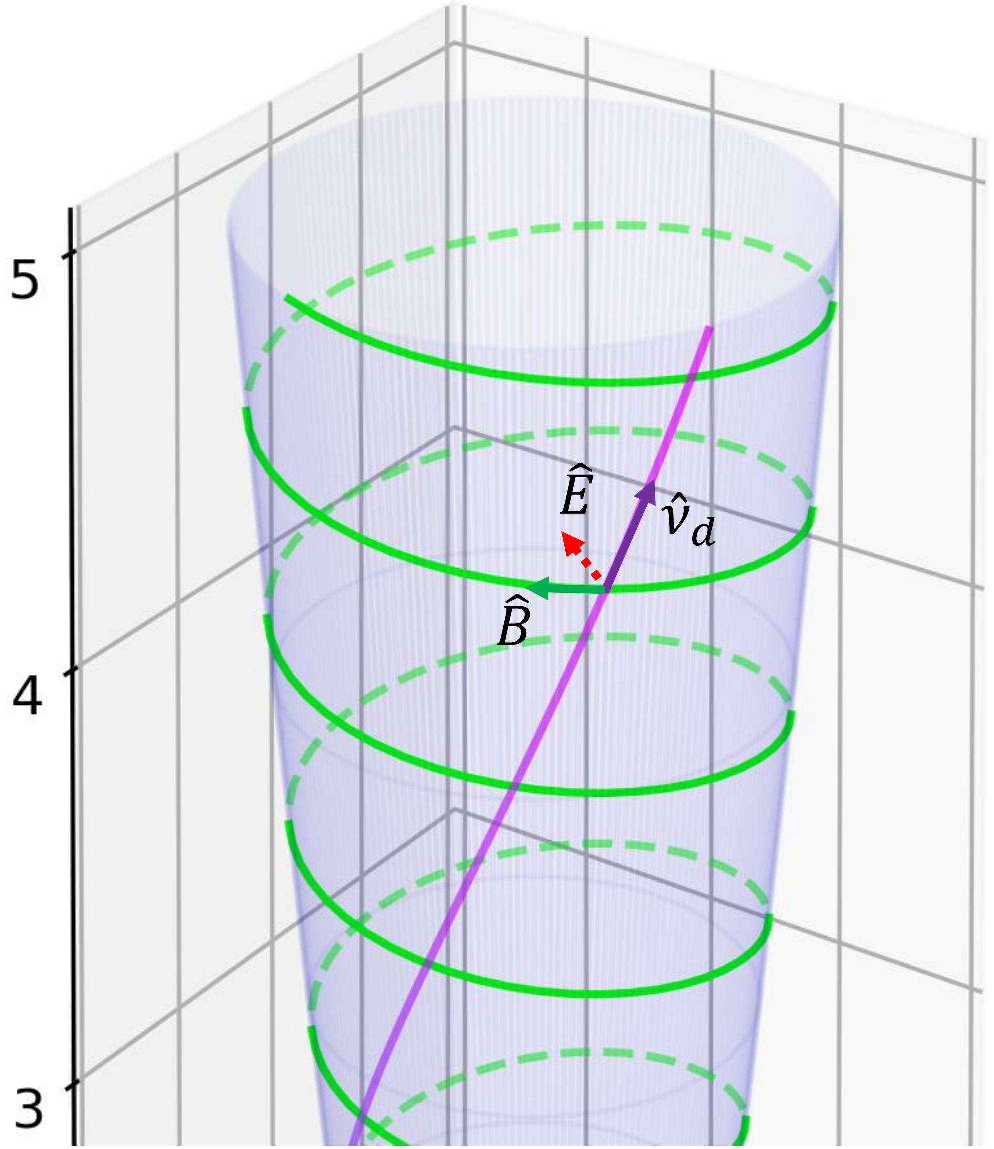
S7: jet dynamics

- The force (I)

$$\mathbf{F}_{\rm L} = \boxed{\overbrace{\underbrace{\mathbf{j}_p \times \mathbf{B}_p}_{1,\hat{\phi}} + \underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_{B_p} \hat{B}_p}_{2,\hat{B}_p}}^{\mathbf{F}_{\rm L,\hat{v}_d}}} + \overbrace{\underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_E \hat{E}}_{3,\hat{E}} + \underbrace{\mathbf{j}_\phi \times \mathbf{B}_p}_{4,\hat{E}} + \underbrace{\rho_{\rm e}\mathbf{E}}_{5,\hat{E}}}^{\mathbf{F}_{\rm L,\hat{E}}}$$

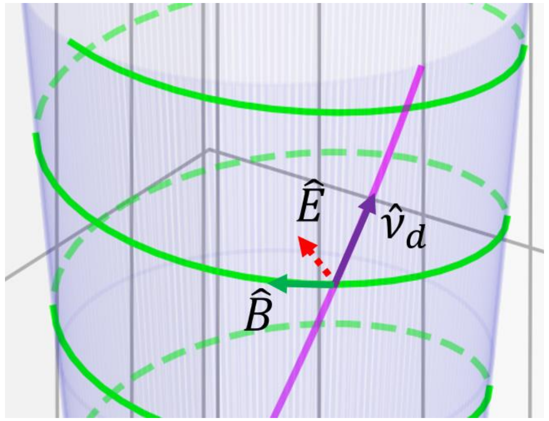
$$\boxed{\mathbf{F}_{\rm L,\hat{v}_d} = \frac{1}{4\pi R^2}\left(\frac{\partial\Phi}{\partial z}\frac{\partial\Psi}{\partial R} - \frac{\partial\Phi}{\partial R}\frac{\partial\Psi}{\partial z}\right)\left[\hat{\phi} + \left(\frac{B_\phi}{B_p}\right)\hat{B}_p\right]}$$

Φ can not be exactly conserved!



S7: jet dynamics

- The force (I) $\mathbf{F}_{\mathrm{L}} = \boxed{\overbrace{\underbrace{\mathbf{j}_p \times \mathbf{B}_p}_{1,\hat{\phi}} + \underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_{B_p} \hat{B}_p}_{2,\hat{B}_p}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{v}}_{\mathrm{d}}}}} + \overbrace{\underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_E \hat{E}}_{3,\hat{E}} + \underbrace{\mathbf{j}_\phi \times \mathbf{B}_p}_{4,\hat{E}} + \underbrace{\rho_{\mathrm{e}} \mathbf{E}}_{5,\hat{E}}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{E}}}}$



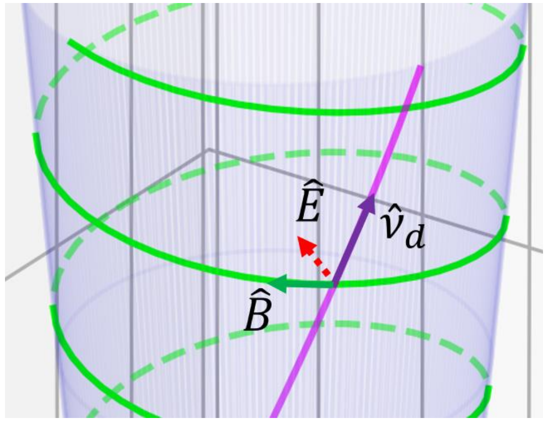
$$\mathbf{F}_{\mathrm{L},\hat{\mathrm{v}}_{\mathrm{d}}} = \frac{1}{4\pi R^2}\left(\frac{\partial\Phi}{\partial z}\frac{\partial\Psi}{\partial R} - \frac{\partial\Phi}{\partial R}\frac{\partial\Psi}{\partial z}\right)\left[\hat{\phi} + \left(\frac{B_\phi}{B_p}\right)\hat{B}_p\right]$$

the following conditions equivalent: 1) The Lorentz force vanishes in the magnetic stream surface (i.e. the force-free condition applies in the surface, $\mathbf{F}_{\mathrm{L},\hat{\mathrm{v}}_{\mathrm{d}}} = 0$); 2) The poloidal current density, velocity, and magnetic field are parallel to each other $\mathbf{j}_p = \Phi'\mathbf{B}_p/4\pi$; 3) The current flows in the magnetic stream surface; 4) Φ conserves along a magnetic field line, and thus is a function of Ψ only. Releasing the force-free assumption (considering the inertia of plasma) requires that the above conditions are broken in order to accelerate the plasma fluid. As a result, these conditions apply approximately in the limit of highly magnetized jet (see below and e.g., Li et al. 1992).

S7: jet dynamics

- The force (I)

$$\mathbf{F}_{\mathrm{L}} = \overbrace{\underbrace{\mathbf{j}_p \times \mathbf{B}_p}_{1,\hat{\phi}} + \underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_{B_p} \hat{B}_p}_{2,\hat{B}_p}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{v}}_{\mathrm{d}}}} + \boxed{\overbrace{\underbrace{(\mathbf{j}_p \times \mathbf{B}_\phi)_E \hat{E}}_{3,\hat{E}} + \underbrace{\mathbf{j}_\phi \times \mathbf{B}_p}_{4,\hat{E}} + \underbrace{\rho_{\mathrm{e}} \mathbf{E}}_{5,\hat{E}}}^{\mathbf{F}_{\mathrm{L},\hat{\mathrm{E}}}}}$$



$$F_{\mathrm{L},\hat{\mathrm{E}}} = \frac{B_p}{R} \left\{ \underbrace{\frac{\partial^2 \Psi}{\partial R^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{R}\frac{\partial \Psi}{\partial R}}_{4} + \underbrace{\Phi \frac{4\pi \mathbf{j}_p \cdot \mathbf{B}_p}{B_p^2}}_{3} \underbrace{- \Omega^2 R^2 \left[\frac{\partial^2 \Psi}{\partial R^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{\Omega'}{\Omega} \left(\left(\frac{\partial \Psi}{\partial R}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2 \right) \right]}_{5} \right\}$$

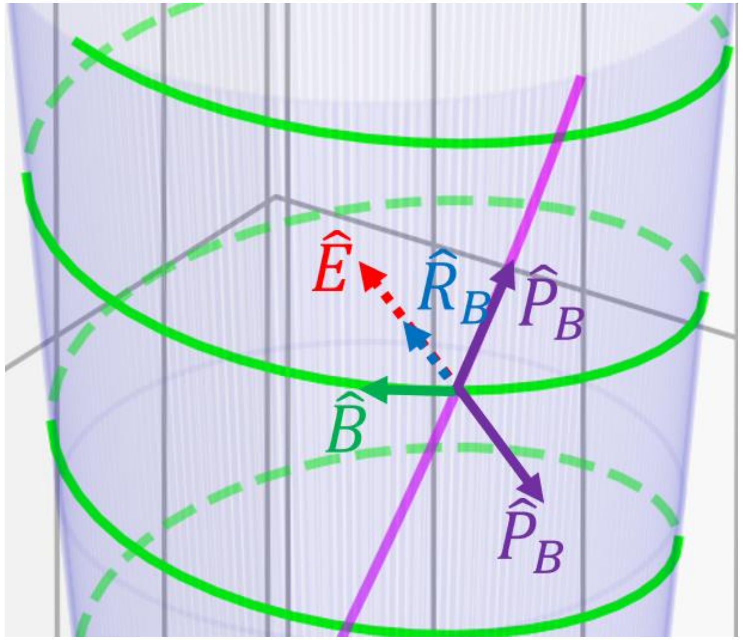
$$\boxed{\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial \Psi}{\partial \theta}}$$

z-pinch

$$+\boxed{\Phi' \Phi} - \boxed{\left\{ \frac{\Omega'}{\Omega} \left[\left(\frac{\partial \Psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right)^2 \right] + \frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \Psi}{\partial \theta} \right\} (\Omega r \sin\theta)^2} = 0$$

"centripetal force" for plasma rotation and self-collimation ↔ highly magnetized jet ↔ self-balance

S7: jet dynamics



- The force (II)
  - magnetic pressure gradient force
  - magnetic tension force
  - electric force

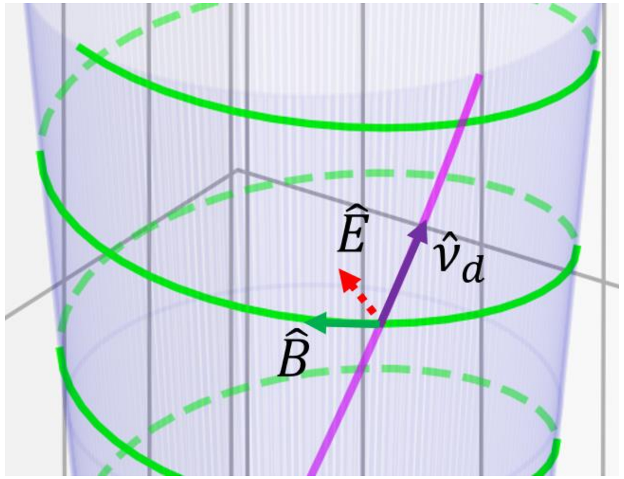
$$\mathbf{F}_{\rm L} = \frac{1}{4\pi}\left(\mathbf{B}\cdot\nabla\right)\mathbf{B} - \nabla\left(\frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} = -\frac{B^2}{4\pi}\frac{\hat{R}_B}{R_B} - \nabla_\perp\left(\frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\left(\nabla\cdot\mathbf{E}\right)\mathbf{E}$$

acceleration: magnetic pressure gradient force

S7: jet dynamics & flow velocity

- cold plasma

$$\nabla \cdot (\rho \mathbf{u}) = 0$$
$$\rho \left(\mathbf{u} \cdot \nabla\right) \mathbf{u} = \rho_{\mathrm{e}} \mathbf{E} + \mathbf{j} \times \mathbf{B}$$
$$\mathbf{v} = \Omega r \sin\theta \hat{\phi} + \kappa \mathbf{B}$$
$$\mathbf{E} = -\Omega \nabla \Psi = -\Omega r \sin\theta \hat{\phi} \times \mathbf{B}$$



conserved qualities

$$4\pi\rho\Gamma\kappa = \eta\left(\Psi\right)$$
$$\Gamma - \frac{B_\phi \Omega R}{\eta} = \mathcal{E}\left(\Psi\right)$$
$$R\Gamma v_\phi - \frac{R B_\phi}{\eta} = \mathcal{L}\left(\Psi\right)$$

magnetization parameter

$$\sigma = \frac{-B_\phi \Omega R}{\eta\Gamma} = \frac{\mathcal{E} - \Gamma}{\Gamma}$$
$$\Gamma\left(\sigma + 1\right) = \mathcal{E}$$

S7: jet dynamics & flow velocity

- Φ approximately conserved for highly magnetically dominated jet flow

$$\Phi = B_\phi R = -\frac{\eta\mathcal{E}}{\Omega}\frac{\sigma}{\sigma+1} \xlongequal{\sigma\gg 1} -\frac{\eta\mathcal{E}}{\Omega}$$

(force-free condition applies)

- However, an exact conservation of Φ prevents any acceleration!

$$\Gamma = \mathcal{E} + \frac{\Omega\Phi}{\eta}$$

S7: jet dynamics & flow velocity

- cold plasma velocity ($r_B = B_\phi / B_p$ and $\varpi = \Omega R$)

$$\Gamma = \frac{\sqrt{r_B^2 \varpi^2 \left(1 + r_B^2 - \varpi^2\right)\left(-1 + \varepsilon^2 + \varpi^2\right)} - \varepsilon \left(1 + r_B^2 - \varpi^2\right)}{\left(1 + r_B^2 - \varpi^2\right)\left(\varpi^2 - 1\right)}$$

- drift velocity

$$\Gamma_{\mathrm{d}} = \sqrt{\frac{1 + r_B^2}{1 + r_B^2 - \varpi^2}}$$

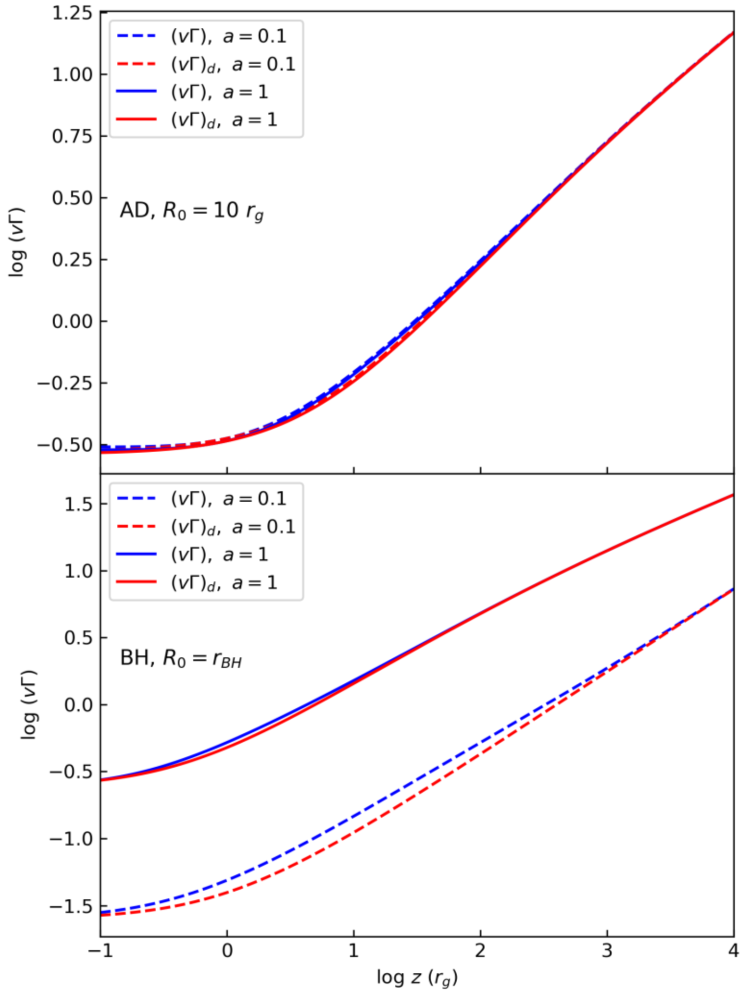
- the deviation $D_{\mathrm{fd}} \equiv \frac{(v\Gamma)^2 - (v_{\mathrm{d}}\Gamma_{\mathrm{d}})^2}{(v_{\mathrm{d}}\Gamma_{\mathrm{d}})^2} = \frac{\left(1 + r_B^2 - \varpi^2\right)\left(\varpi^2 - 1\right) - 2\varepsilon\sqrt{r_B^2 \varpi^2 \left(1 + r_B^2 - \varpi^2\right)\left(-1 + \varepsilon^2 + \varpi^2\right)} + \varepsilon^2 \left[1 - \varpi^2 + r_B^2 \left(1 + \varpi^2\right)\right]}{\varpi^2 \left(\varpi^2 - 1\right)^2}$

$$D_{\mathrm{fd}} \equiv \frac{(v\Gamma)^2 - (v_{\mathrm{d}}\Gamma_{\mathrm{d}})^2}{(v_{\mathrm{d}}\Gamma_{\mathrm{d}})^2} \ll 1 \qquad (\Omega R \gg 1 \text{ or } \Omega \mathrm{R} \ll 1)$$

S7: jet dynamics & flow velocity

- cold plasma velocity matches the drift velocity very well

$$\mathbf{v}_{\rm d} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$



Why the plasma fluid velocity can be represented by a pure electromagnetic field quality - drift velocity: 1) The plasma flow is highly magnetized so that it is dynamically unimportant and almost cannot affect the electromagnetic field configuration; 2) The ideal MHD condition indicates that the motion of the plasma fluid is governed by the electromagnetic field configuration (freezing effect); 3) The electromagnetic field configuration is self-consistently determined because the plasma can supply currents and charges as needed to support the electromagnetic field.

S7: jet dynamics & flow density

- given an available mass flux per magnetic flux η that still satisfies a highly magnetically dominated condition
- ρ is proper density, $\rho_{\rm l}$ measured in lab frame

$$\rho = \frac{\eta}{4\pi}\frac{B_p}{u_p} \simeq \frac{\eta\Psi}{2\pi\Omega^2}\frac{\sqrt{1+(\Omega R)^2}}{R^4} \simeq \frac{\eta}{4\pi\Omega^2}\frac{B}{R^2}$$

$$\rho_{\rm l} = \Gamma\rho \simeq \frac{\eta}{8\pi\Psi\Omega^2}B^2,$$

S8: black hole jet

- pseudo-Newtonian potential to measure GR effect (Artemova et al. 1996)

$$g_{\rm BH} = -\frac{M}{r^{2-\gamma}\left(r - r_{+}\right)^{\gamma}} \qquad \gamma = r_{\rm ISCO}/r_{+} - 1$$

- motion equation

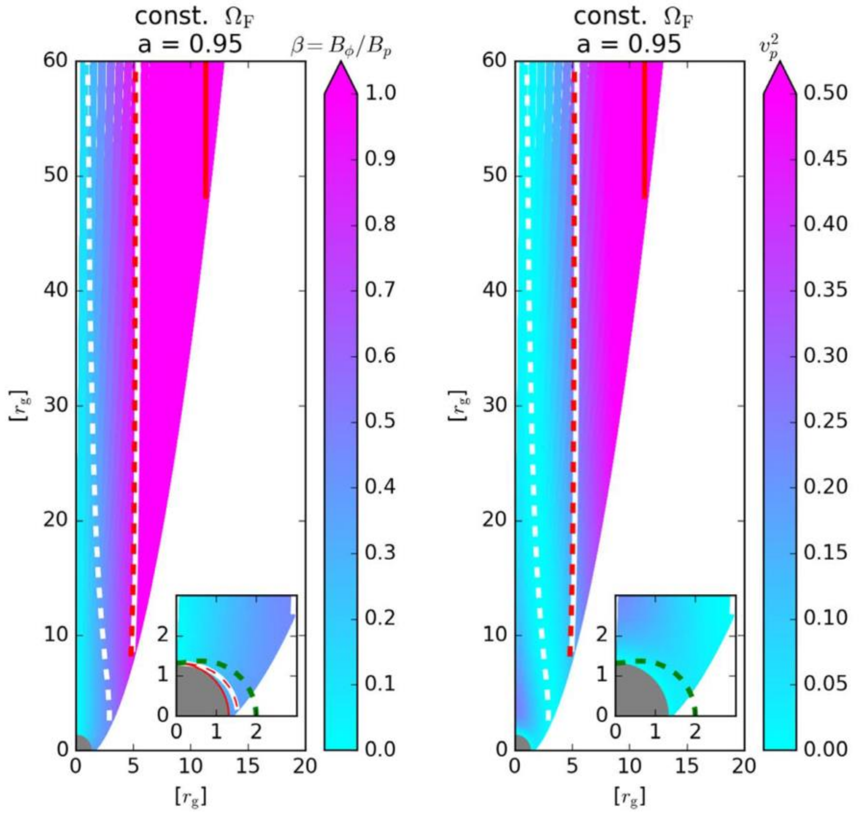
$$\rho\left(\mathbf{u}\cdot\bigtriangledown\right)\mathbf{u} = \rho_{\rm e}\mathbf{E} + \mathbf{j}\times\mathbf{B} + \rho g_{\rm BH}\hat{\mathbf{r}}$$

- magnetic force potential ($\sigma = U_{\rm B}/\rho$)

$$|g_{\rm B}| = \sigma\left(\frac{B^2}{8\pi}\right)^{-1}|(\mathbf{j}\times\mathbf{B})_r| = 8\nu\sigma\Omega^2 r\frac{1}{\left(1+B_\phi^2/B_p^2\right)\left(\nu^2+T'^2/T^2\right)} \approx 4\nu\sigma\Omega^2 r\frac{1-\cos\theta}{\left(1+B_\phi^2/B_p^2\right)}$$

$$a = 0.998,\ \sigma = 10,\ \theta = \pi/4 \text{ and } \nu = 3/4 \Longrightarrow r_{\rm cb} \approx 1.4\ r_{\rm g}$$

stagnation surface



Pu & Takahashi 2020

S8: black hole jet

- BH angular velocity ($f_\Omega \approx 0.3 - 0.5$ McKinney & Narayan 2007)

$$\Omega = f_\Omega \Omega_{\rm BH} = f_\Omega \frac{a}{2\left(1+\sqrt{1-a^2}\right) r_{\rm g}}$$

- constraint

$$T\left(\theta_0\right) \geq C_2 \theta^{2-\nu}\left(\frac{2 v \Gamma}{f_\Omega}\right)^\nu,$$

$$a \geq a_{\min} = C_2^{1/\nu} \theta^{2/\nu-1}\left(\frac{2 v \Gamma}{f_\Omega}\right),$$

$$M \geq \frac{f_\Omega a_{\min} R}{2\left(1+\sqrt{1-a_{\min}^2}\right) v \Gamma},$$

$$r_0 = r_+ = (1+\sqrt{1-a^2}) r_{\rm g}.$$

S8: black hole jet

- AD angular velocity

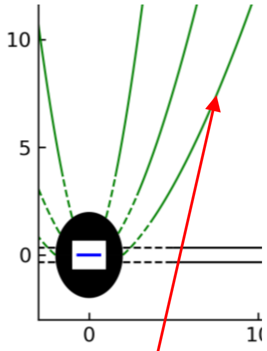
$$\Omega \lesssim \Omega_{\mathrm{K}} = \frac{1}{r_{\mathrm{g}}\left[\left(r_0/r_{\mathrm{g}}\right)^{3/2} + a\right]} \leq \frac{1}{r_{\mathrm{g}}\left(r_0/r_{\mathrm{g}}\right)^{3/2}}$$

- constraint

$$\begin{aligned} \theta_0 &= \pi/2, \\ r_0 &= C_2^{1/\nu} r \theta^{2/\nu}, \\ M &\geq C_2^{3/\nu}\left(v\Gamma\right)^2 r \theta^{6/\nu-2}, \\ \frac{1}{\left(r_{\mathrm{ISCO}}/r_{\mathrm{g}}\right)^{3/2}} &\geq C_2^{3/\nu}\left(v\Gamma\right)^3 \theta^{6/\nu-3}. \end{aligned}$$

S8: black hole jet

- jet properties



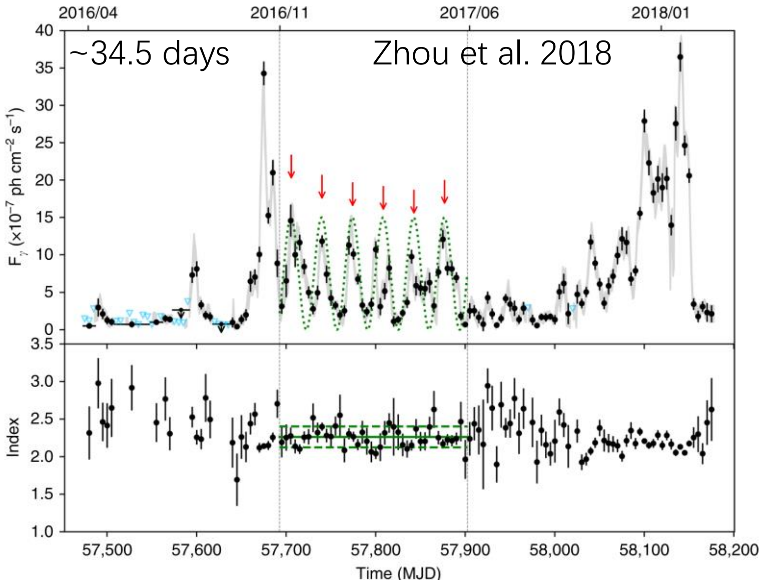
$$\frac{R_{\rm out}}{r_{\rm g}} = C_2^{-1/2}\left(1+\sqrt{1-a^2}\right)^{\nu/2}\left(\frac{z}{r_{\rm g}}\right)^{1-\nu/2},$$

$$V_{1,\theta\ll 1,{\rm out}} = C_2^{-1/2}\left(\frac{f_\Omega}{1/2}\right)\frac{a}{4\left(1+\sqrt{1-a^2}\right)^{1-\nu/2}}\left(\frac{z}{r_{\rm g}}\right)^{1-\nu/2},$$

$$V_{2,\theta\ll 1,{\rm out}} = \frac{2}{\sqrt{2-\nu}}C_2^{1/2}\left(1+\sqrt{1-a^2}\right)^{-\nu/2}\left(\frac{z}{r_{\rm g}}\right)^{\nu/2},$$

$$P_1 = 3.44\frac{1+\sqrt{1-a^2}}{a}\left(\frac{1/2}{f_\Omega}\right)\left(\frac{M}{10^8 M_\odot}\right)\ {\rm hr},$$

$$(v\Gamma)^2 B' = (\Omega R)^2 B_p = \frac{2}{\nu^2+C_1^2}(\Omega R_0)^2 B_{p,0} = \frac{2}{\nu^2+C_1^2}\left(\frac{f_\Omega}{1/2}\right)^2\frac{a^2 B_{p,0}}{16},$$



$$P \approx \left(1+\Gamma^2\right) P_1$$

days - years

AGN: period, oscillation

(Ackermann et al. 2015; Penil et al. 2020
Lister et al. 2013; Walker et al. 2018)

at distance $z = 10^3\ r_{\rm g}$, one has $R_{\rm out} = 109\ r_{\rm g}$, $V_{1,\theta\ll 1,{\rm out}} = 27.3$, $V_{2,\theta\ll 1,{\rm out}} = 16.4$ and $(v\Gamma)_{\theta\ll 1,{\rm out}} = 14.1$

$(v\Gamma)_{\rm out} = 1$ is located at $z = 4.99\ r_{\rm g}$ for $a = 1$, at $z = 67.4\ r_{\rm g}$ for $a = 0.3$, at $z = 800\ r_{\rm g}$ for $a = 0.1$

S8: black hole jet

- maximum velocity

$$(v\Gamma)_{\rm CCS} = V_{1,\theta\ll 1,{\rm crs}}/\sqrt{2} = \frac{1}{(2-\nu)^{1/4}}\left[\frac{a}{4\left(1+\sqrt{1-a^2}\right)}\left(\frac{f_\Omega}{1/2}\right)\frac{z}{r_{\rm g}}\right]^{1/2}$$

$$\Omega R^2 z^{-1} = \frac{2}{\sqrt{2-\nu}}$$

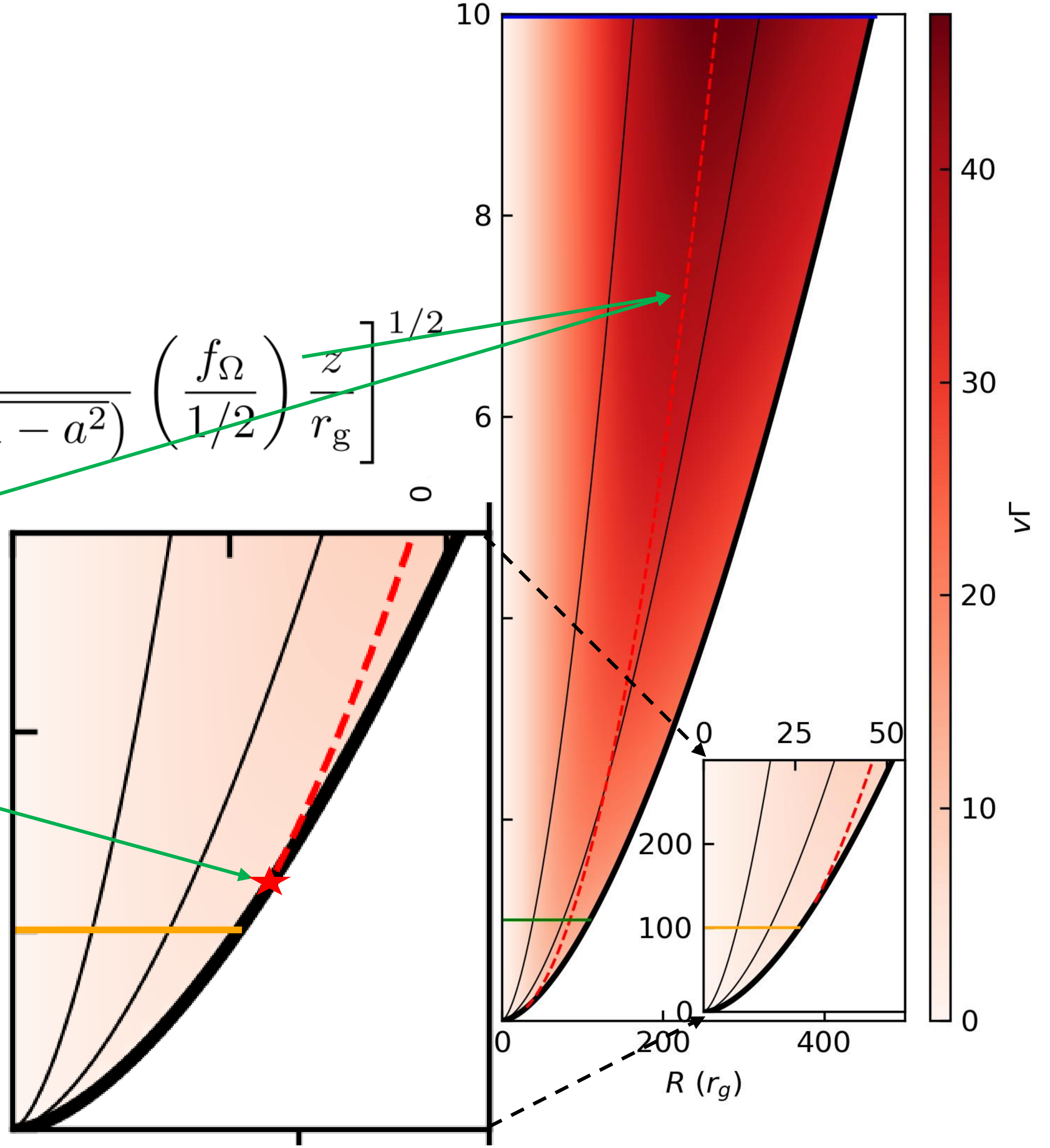
- CCS vs. outmost

$$\frac{z_{\rm crs}}{r_{\rm g}} = \left(1+\sqrt{1-a^2}\right)\left[\frac{2}{\sqrt{2-\nu}}\left(\frac{1/2}{f_\Omega}\right)\frac{4C_2}{a}\right]^{1/(1-\nu)}$$

- GRB $\boxed{(v\Gamma) \lesssim 1/\theta\,?}$ Lithwick & Sari 2001

$$z = 10^5\ r_{\rm g} \qquad (v\Gamma)_{\rm CCS} = 150$$

$$\theta_{\rm op} = 2\sin^{-1}(R_{\rm out}/z) = 2.22^\circ$$



S8: black hole jet

- jet power (BZ)

$$P_{\rm jet} \sim 2\times 10^{45}\frac{2}{\nu^2+C_1^2}\left(1+\sqrt{1-a^2}\right)^2 a^2\left(\frac{f_\Omega}{1/2}\right)^2\left(\frac{M}{10^8 M_\odot}\right)^2\left(\frac{B_{p,0}}{10^5\mathrm{Gs}}\right)^2 \ \mathrm{erg\ s^{-1}}$$

$$\sim 2\times 10^{51}\frac{2}{\nu^2+C_1^2}\left(1+\sqrt{1-a^2}\right)^2 a^2\left(\frac{f_\Omega}{1/2}\right)^2\left(\frac{M}{10 M_\odot}\right)^2\left(\frac{B_{p,0}}{10^{15}\mathrm{Gs}}\right)^2 \ \mathrm{erg\ s^{-1}}$$

- jet current (3C 303)

$$J=\sqrt{P_{\rm jet}}\approx 5.8\times 10^{17}\sqrt{P_{44}}\ \mathrm{A}$$

$\sim 3.9\times 10^{18}\ \mathrm{A}$ Kronberg et al. 2011

$\sim 1.0\times 10^{46}\ \mathrm{erg\ s^{-1}}$ Zhang et al. 2018

- potential difference

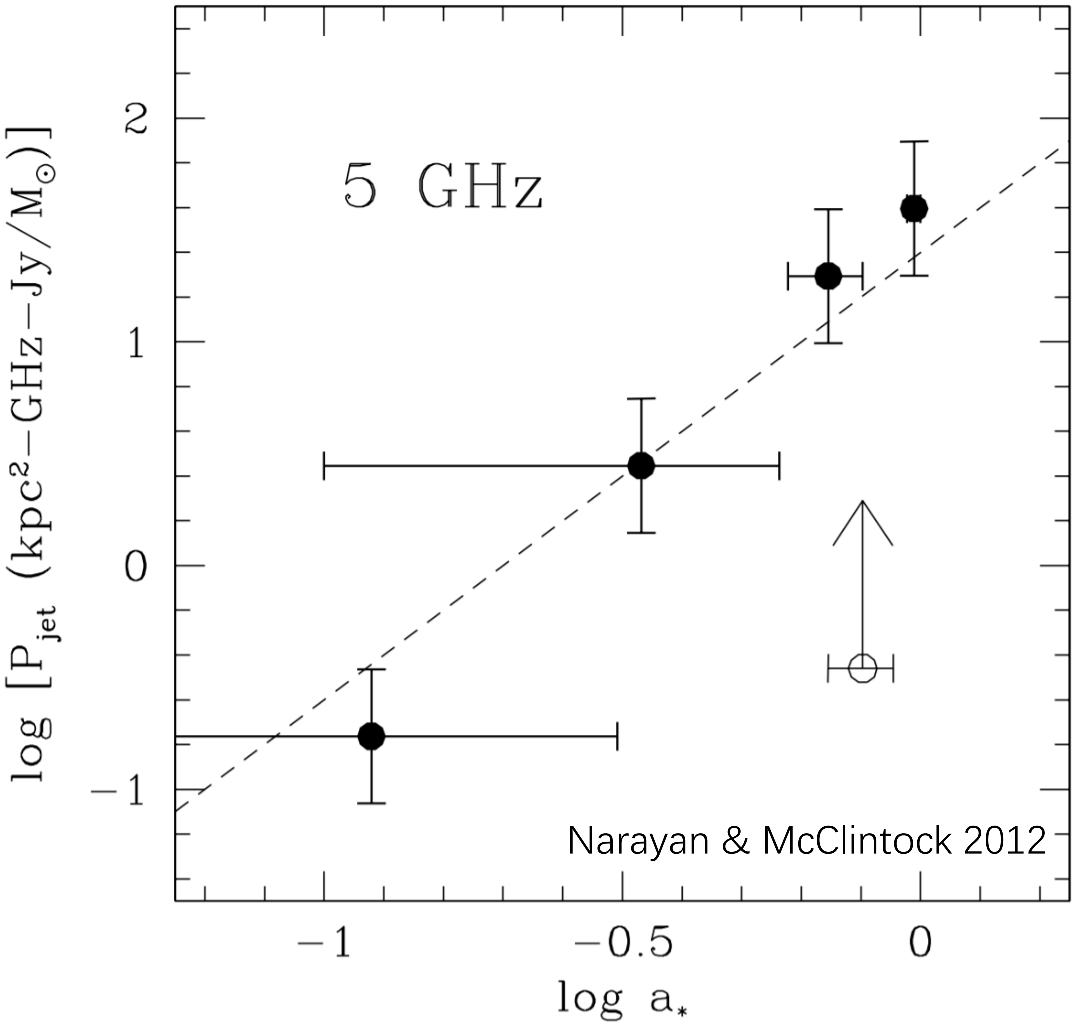
$$\Delta V=\sqrt{P_{\rm jet}}\approx 1.7\times 10^{19}\sqrt{P_{44}}\ \mathrm{volts}$$

UHE cosmic rays
>1 EeV Kotera & Olinto 2011
highest ~10^{21} eV Abraham et al. 2010

S8: black hole jet

- jet power (BZ)

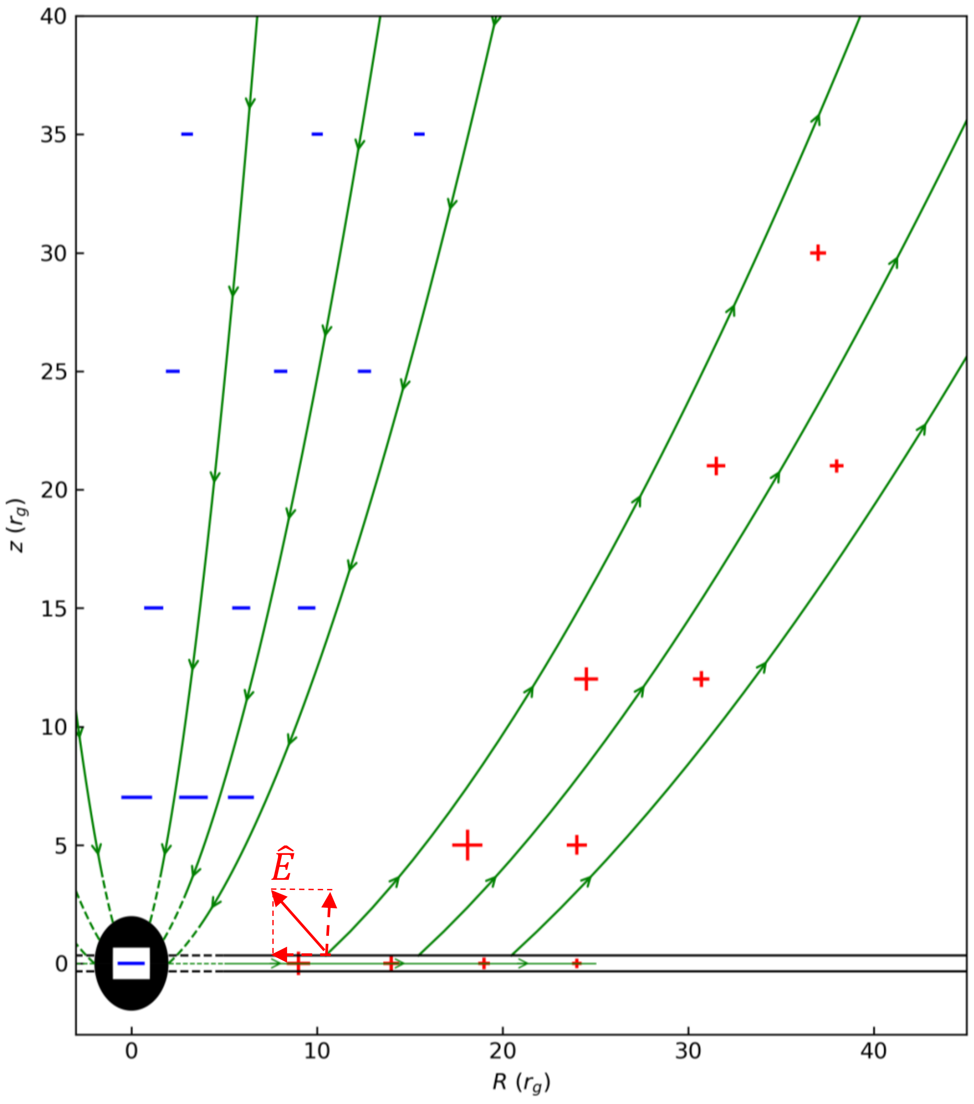
$$P_{\rm jet} \propto a^2$$



S9: CO/AD as boundary

- AD surface density
  - Stokes's theorem to $\mathbf{j} = \nabla \times \mathbf{B}/4\pi$
  - Gauss's theorem to $\rho_\mathrm{e} = \nabla \cdot \mathbf{E}/4\pi$

$$J_\phi^\mathrm{s}(R) = \frac{B_R(R)}{2\pi} = \frac{C_1}{2\pi} R^{\nu-2},$$
$$J_R^\mathrm{s}(R) = -\frac{B_\phi(R)}{2\pi} = \frac{\Omega R}{\pi} R^{\nu-2},$$
$$\rho_\mathrm{e}^\mathrm{s}(R) = -\frac{C_1 B_\phi(R)}{4\pi} = \frac{C_1 \Omega R}{2\pi} R^{\nu-2}.$$



S9: CO/AD as boundary

- CO total charge

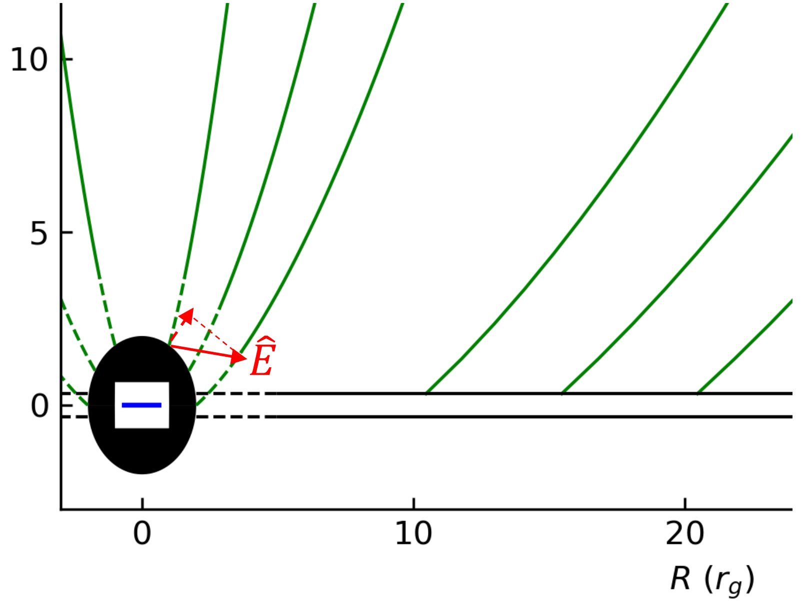
$$Q = \frac{1}{4\pi}\oint \mathbf{E}\cdot d\mathbf{S} = \frac{-\nu C_1 \Omega r_0^3 B_{p,0}}{(1+\nu)(2-\nu)\sqrt{\nu^2+C_1^2}} = -Sg\,(\mathbf{\Omega}\cdot\mathbf{B})\,\frac{\nu C_1 r_0 \sqrt{P_{\rm jet}}}{(1+\nu)(2-\nu)} = -\frac{Sg\,(\mathbf{\Omega}\cdot\mathbf{B})\,\nu C_1 f_\Omega a F_{\rm B}}{4\pi(1+\nu)(2-\nu)}$$

- BH/AD charged vs. RL/RQ in AGN
- BH charge

Kerr – Newman metric $r_{\rm Q}^2 + a^2 \leq 1$

$$r_{\rm Q} = \frac{Q}{M} \approx \frac{r_0}{r_+}\sqrt{\frac{P_{\rm jet}}{3.63\times10^{59}\ {\rm erg\ s^{-1}}}} \ll 1$$

$$\Delta t \sim Q/J \sim 10 M_8\ {\rm min}$$



S10: stability

- marginally stable to the kink instability (Tomimatsu et al. 2001)

$$B_\phi \approx -\Omega R B_p$$

- spine-layer structure may be naturally stabilized
  (e.g., Mizuno et al. 2007; Hardee 2007)

- M87

S11. conclusion & summary

In this paper, starting from the first principles, we study an analytical solution of a magnetized jet/wind flow. The explicit expression of the solution makes it easy for further developments (e.g. adding radiative processes) and to directly compare with the observational data. Our findings can be summarized as follows.

1. Through separating the force-free equation of a jet/wind plasma flow into the non-rotating and rotating parts, we found that each of the two equations can be solved analytically and that the two solutions match each other very well (in the regimes either $\theta \ll 1$ or $\theta \to \pi/2$, and either non-relativistic or relativistic). Therefore, we have a general approximate solution of a highly magnetized jet.

2. An ordered rotating magnetic field is the indispensable ingredient to globally launch, accelerate, and collimate a relativistic jet. The magnetic stream function $\Psi = C_2 r^\nu \sin^2\theta \; {}_2F_1\left(1 - \frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}, 2, \sin^2\theta\right)$ (with $0 \le \nu \le 2$ a free parameter) controls the poloidal magnetic field configuration being a general parabola, while the toroidal magnetic field is determined by $\Phi = -2\Omega\Psi$ with an angular velocity $\Omega = \alpha\Psi^\lambda$. The resulting helical magnetic field is dominated by the poloidal component within the ACS, and by the toroidal component outside. The large scale jet configuration can be used to constrain the foot-point locations of the magnetic fields.

S11. conclusion & summary

3. For a highly magnetized jet/wind flow, the drift velocity matches the cold plasma velocity very well, which is almost always perpendicular to the magnetic field and therefore also forms a helical structure. Acceleration from non-relativistic through relativistic regimes is described, which is divided into three stages in the case of $\nu < 1$: 1) within the ACS, where the toroidal velocity dominates and the four velocity reads $v\Gamma = \Omega R$; 2) outside the ACS but within the CCS, where the poloidal velocity dominates and $v\Gamma = \Omega R$; and 3) outside the CCS, where the dominant poloidal velocity follow $v\Gamma \approx 2/\left(\theta\sqrt{2-\nu}\right)$ due to a causality constraint. The acceleration in the case of $\nu \geq 1$ only has the former two stages and therefore always follows $v\Gamma = \Omega R$. The large scale jet velocity can be used to constrain angular velocity, and hence, the spin of the central BH.

4. The energy transportation of a magnetized jet/wind is dominated by the Poynting flux. The jet power is determined by the angular velocity and the magnetic flux, i.e. $P_{\rm jet} = \Omega^2 F_{\rm B}^2/\left(4\pi^2\left|\lambda+1\right|\right) \approx 2\times 10^{45} a^2 M_8^2 B_5^2$ erg s^{-1} in the case of magnetic fields threading a BH ($\lambda = 0$ and $B_5 = B/10^5{\rm Gs}$).

5. A jet/wind flow has to carry charges so that the electric field force can balance the magnetic field force. The charge density reads $\rho_{\rm e} = -\mathbf{\Omega}\cdot\mathbf{B}/2\pi$ (in the case of magnetic fields threading a CO, $\lambda = 0$). The central rotating CO also has to be charged to globally launch a magnetized (BZ) jet with a total charge $|Q| \approx aF_{\rm B}/8\pi \approx r_0\sqrt{P_{\rm jet}} \approx 2.8\times 10^{20}\left(1+\sqrt{1-a^2}\right) M_8\sqrt{P_{44}}$ C (the BH case, the sign determined by $-Sg\left(\mathbf{\Omega}\cdot\mathbf{B}\right)$, $P_{44} = P_{\rm jet}/10^{44}$ erg s^{-1}). Whether the BH (AD) is charged may account for the RL/RQ dichotomy in AGNs, with the aid of spin of the BH.

S11. conclusion & summary

6. In a jet/wind flow, the toroidal current almost vanishes, while the poloidal current is about $j_p \approx \rho_{\rm e}$. The total current carried by the jet reads $J = \sqrt{|\lambda+1|\, P_{\rm jet}} \approx 5.8 \times 10^{17} \sqrt{P_{44}}$ A $(\lambda = 0)$.

7. The magnetic stream surface is equipotential, within which the magnetic field lines lie, the current streams and the plasma flows. Crossing these surfaces can in principle lead to acceleration of the charged particles, which may be achieved through forming a gap in the polar region. The acceleration is limited by the potential difference between two magnetic stream surfaces $\Delta V = \sqrt{P_{\rm jet}/\,|\lambda+1|} \approx 1.7 \times 10^{19} \sqrt{P_{44}}$ volts $(\lambda = 0)$.

8. Given an available mass flux per magnetic flux (η) that still satisfies a highly magnetically dominated condition, one has approximations of the proper density $\rho \approx \left(\eta/4\pi\Omega^2\right) B/R^2$ and the lab frame density $\rho_{\rm l} = \Gamma\rho \approx \left(\eta/8\pi\Psi\Omega^2\right) B^2$.

9. This approximate solution can (roughly) match known numerical simulation results and interpret most observations of AGN and GRB jets.

S11. conclusion & summary

- Jet launching, acceleration and collimation

numerically vs. analytically

- Core equation solving ("pulsar" equation into two parts)

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta} \rightarrow 0$$

$$+\Phi'\Phi-\left\{\frac{\Omega'}{\Omega}\left[\left(\frac{\partial\Psi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)^2\right]+\underbrace{\frac{\partial^2\Psi}{\partial r^2}+\frac{2}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}}_{\rightarrow 0}\right\}(\Omega r\sin\theta)^2=0$$

- An analytically quantitative 3D jet

(magnetic filed, velocity, density etc)

$$\boxed{\frac{-B_\phi}{B_p}\simeq\frac{v_p}{v_\phi}\simeq\frac{\Omega R}{c}\simeq\frac{v\Gamma}{c}}\quad\boxed{B_p\simeq\frac{2\Psi}{R^2}}\quad\boxed{\rho_1\simeq\frac{\eta}{8\pi\Psi\Omega^2}B^2}$$

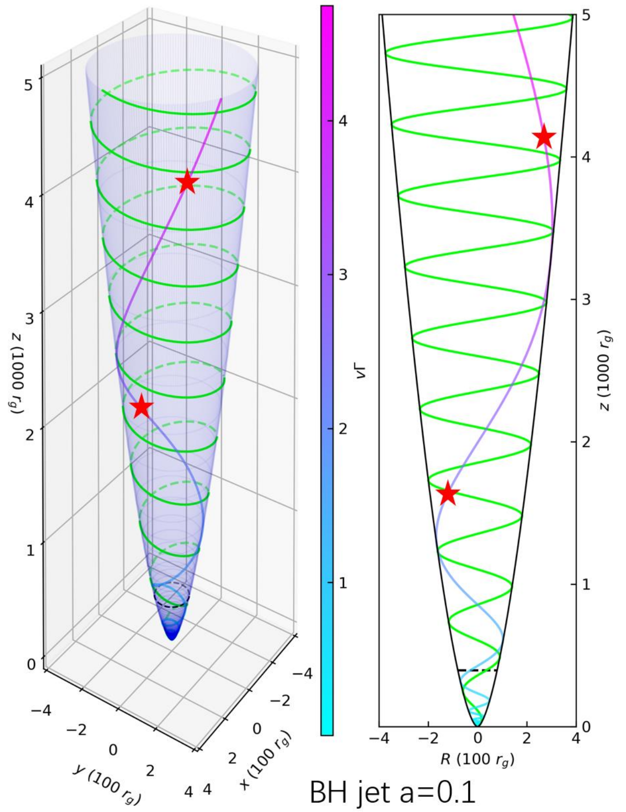
- Match observational and numerical results; predict more phenomena

jet shape configuration, acceleration profile (from non-relativistic to relativistic), polarization pattern, limb-brightening (a hollow jet), periodical signals (a helical jet), stratified jet etc

Jet problem

- seems to be complex MHD system
- a simple electrodynamical problem

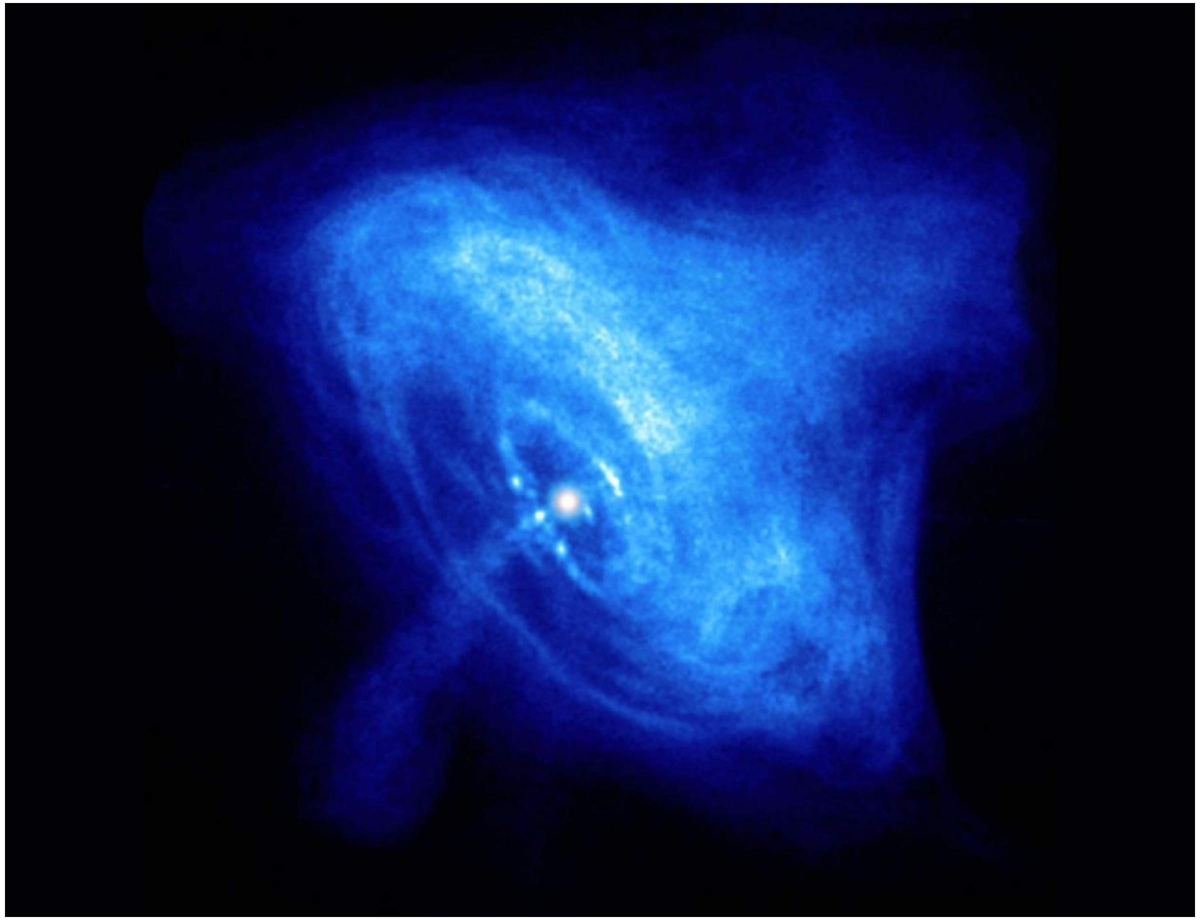
cartoon ⟶ physical



BH jet a=0.1

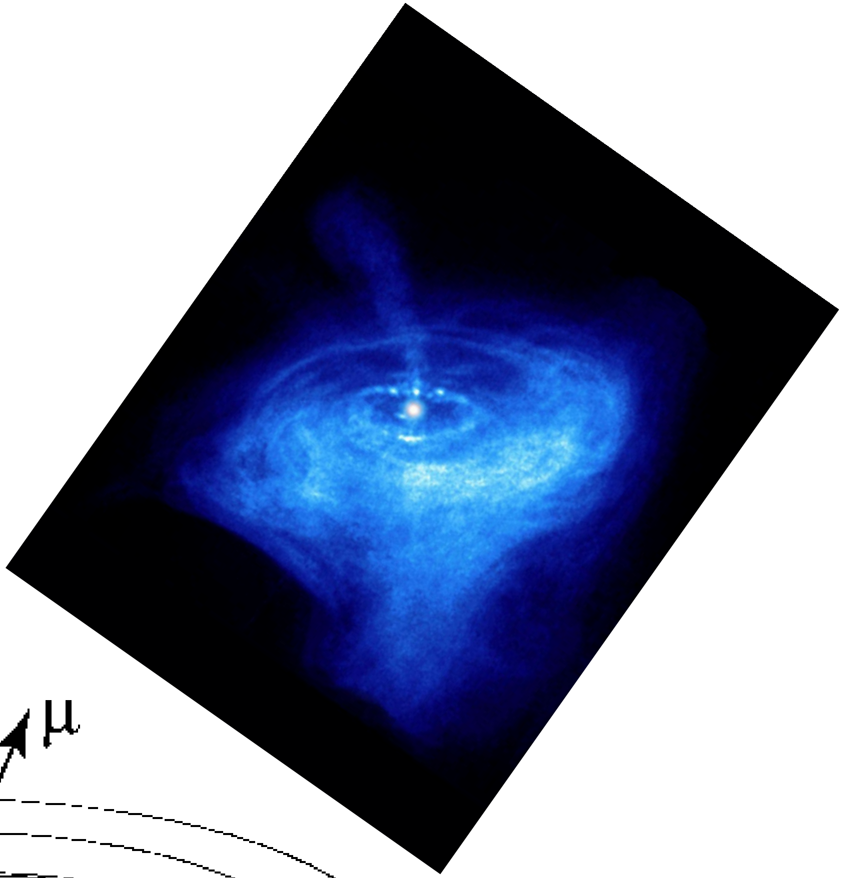
Thanks!

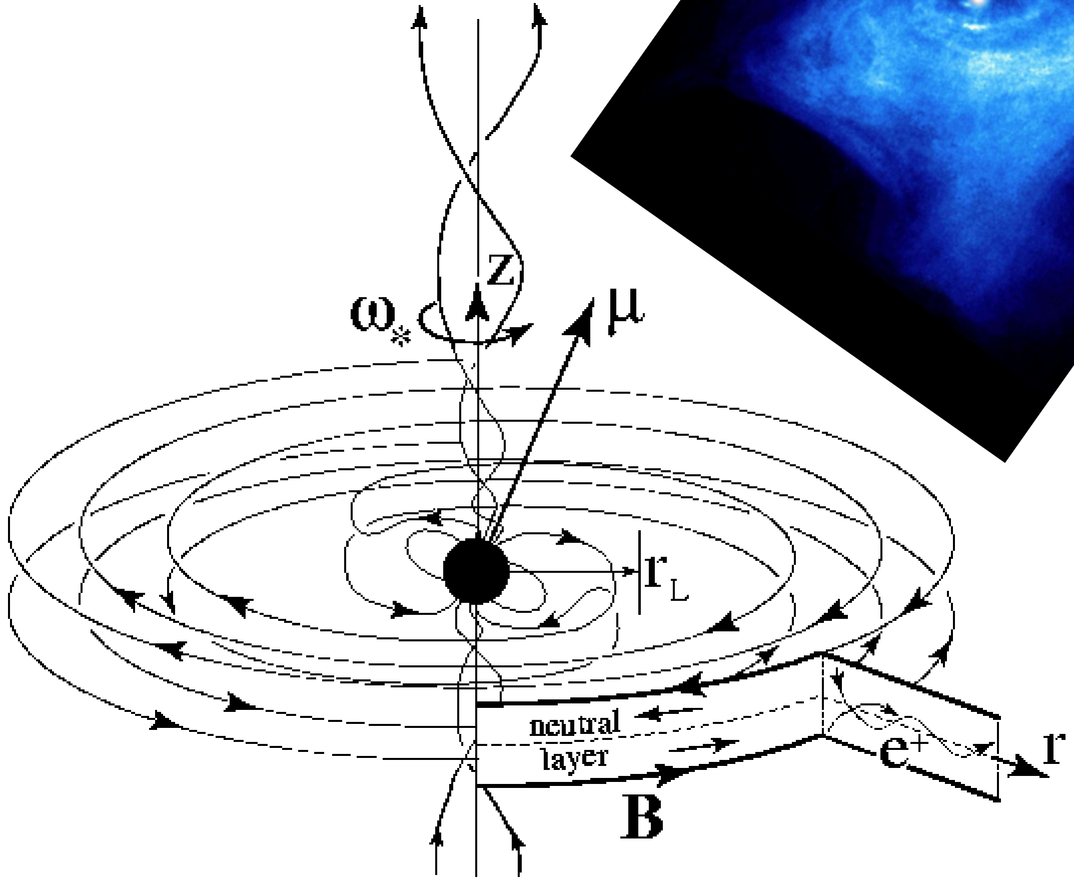
Pulsar Nebula – Crab



Pulsar - misaligned

- self-balanced
- polar magnetic field
- equatorial magnetic field
- “jet-torus” feature

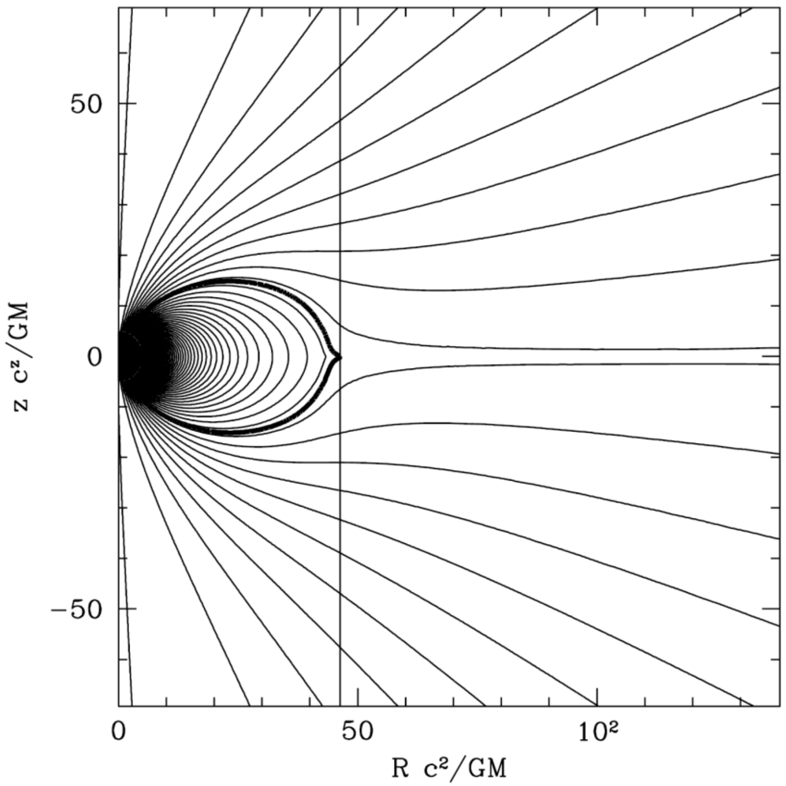


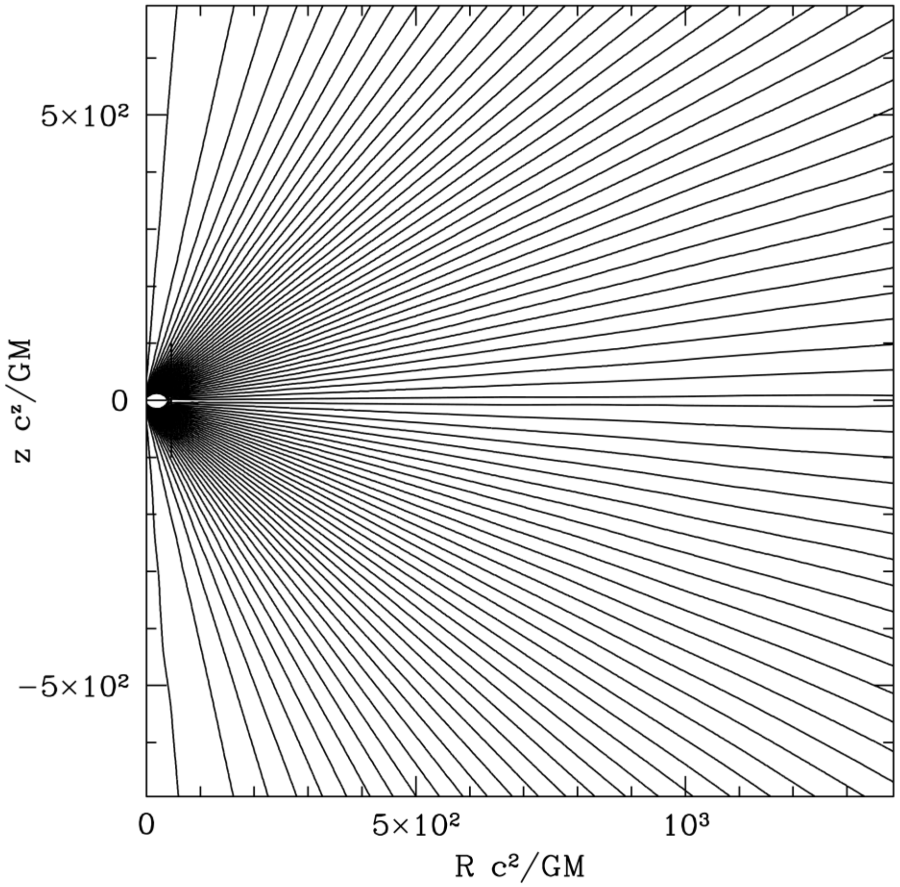


Pulsar - aligned

wind velocity

$$v\Gamma = \Omega R \sim 10^{3-6}$$ Rees & Gunn 1974; Kennel & Coroniti 1984





McKinney 2006

Pulsar – jet - electric potential difference

- Potential difference

$$\Delta V = \int_{l_1}^{l_2} \mathbf{E} \cdot d\mathbf{l}_{\rm E}$$

$$= -\int_{l_1}^{l_2} \Omega \nabla \Psi \cdot d\mathbf{l}_{\rm E} = -\frac{\Omega \Psi}{(\lambda+1)}\Big|_1^2 = -\frac{\Omega F_B}{2\pi(\lambda+1)}\Big|_1^2 = \frac{R B_\phi}{2(\lambda+1)}\Big|_1^2 = \frac{J}{(\lambda+1)}\Big|_1^2$$

- Compact Object (UHE cosmic rays)

$$\Delta V = \sqrt{P_{\rm jet}} \approx 1.7 \times 10^{19} \sqrt{P_{44}}$$

>1 EeV   Kotera & Olinto 2011

highest ~3 * 10^{20} eV   Abraham et al. 2010

- Crab Nebula

$$\Delta V = \sqrt{P_{\rm jet}} \approx 1.7 \times 10^{19} \sqrt{P_{44}} \approx 5 \times 10^{15} \text{ volts}$$

$\gtrsim 10^{37}$ erg s^{-1} Hester 2008

≈ 0.45 PeV   Amenomori et al. 2019