# Cosmic Structure Formation With Analytic Methods

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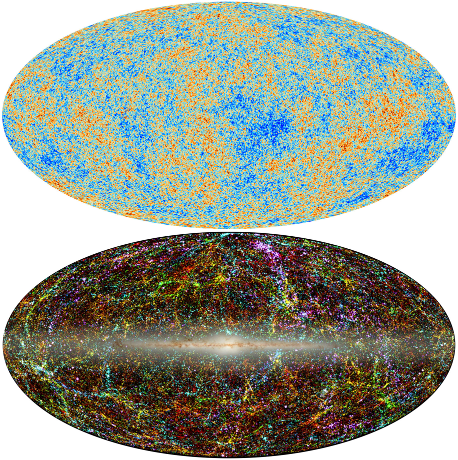
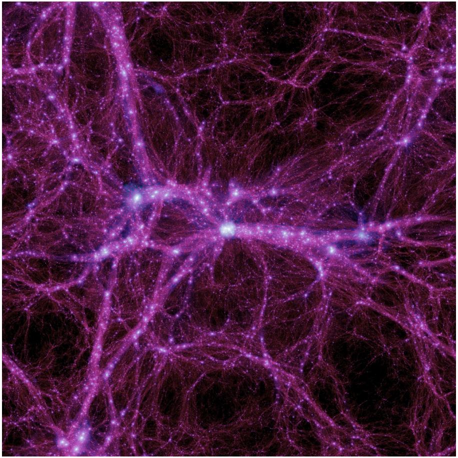
# Cosmic structure formation



Planck, 2-MASS



Boylan-Kolchin et al.

# Problems in cosmic structure formation

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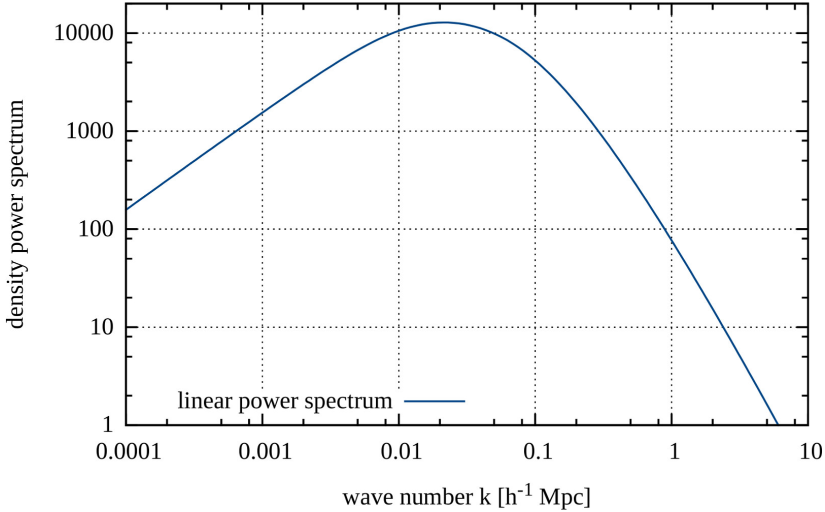

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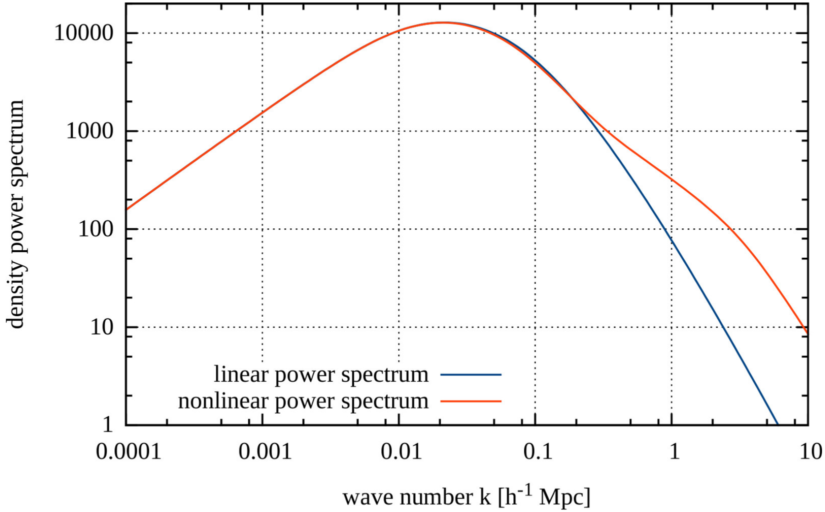

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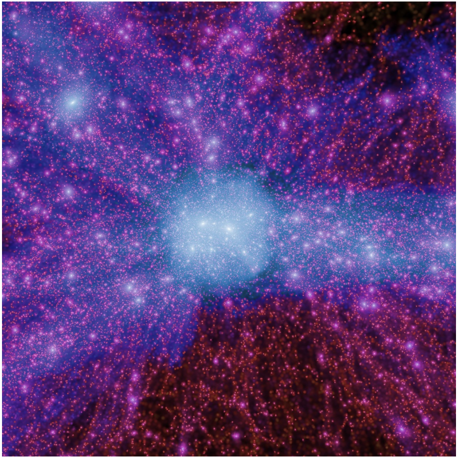
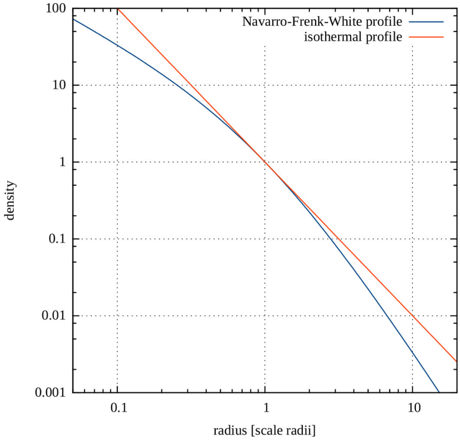


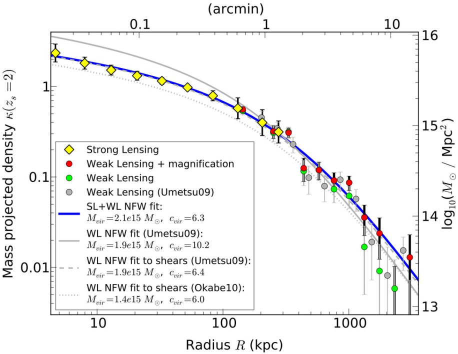
Boylan-Kolchin et al.



# Problems in cosmic structure formation

Abell 2261, CLASH Project

# Problems in cosmic structure formation




Coe et al. 2012

# Cosmological perturbation theory



Conventional:

- Hydrodynamical equations:

$$\dot{\delta} + \vec{\nabla} \cdot \vec{u} = 0$$
$$\dot{\vec{u}} + 2H\vec{u} = -\vec{\nabla}\phi$$
$$\vec{\nabla}^2\phi = 4\pi G\bar{\rho}\delta$$

- But: dark matter is no fluid
- Multiple streams form where shocks would form in a fluid

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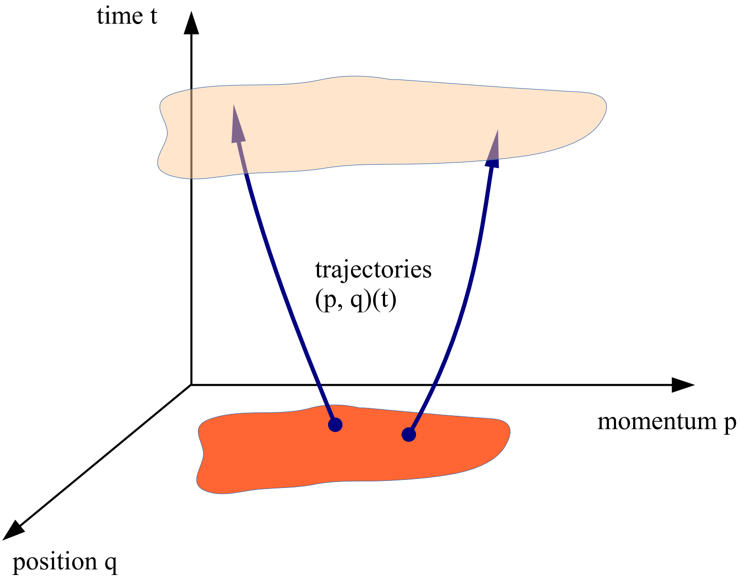
Kinetic Field Theory:

- Non-equilibrium statistics of $N$ classical particle trajectories
- Describe particle ensemble by partition sum (generating functional) $Z$
- Derive statistical properties by functional derivatives

# Phase-space trajectories





# Phase-space trajectories



- Classical particles follow Hamiltonian equations of motion,

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}\,, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}\,, \quad x = (q,p)$$

- Trajectories are described by (retarded) Green's function $G_{\mathrm{R}}(t,t')$,

$$\bar{x}(t) = \underbrace{G_{\mathrm{R}}(t,0)x^{(\mathrm{i})}}_{\text{free motion}} - \underbrace{\int_0^t G_{\mathrm{R}}(t,t')\nabla v(t')\mathrm{d}t'}_{\text{interaction}}$$

# Principle



Non-equilibrium, statistical theory for classical degrees of freedom:

$$Z = \int \mathcal{D}\psi \, P(\psi)$$

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$$= \int \mathrm{d}x \int \mathrm{d}\Gamma^{(\mathrm{i})}\, \delta_{\mathrm{D}}\left[x - \bar{x}(x^{(\mathrm{i})})\right] \mathrm{e}^{\mathrm{i}\langle J,x\rangle}$$

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&= \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{I}}} Z_0[J] \\
\langle\rho(1)\rho(2)\rangle &= \frac{\delta}{\mathrm{i}\delta J(1)} \frac{\delta}{\mathrm{i}\delta J(2)} Z[J]
\end{aligned}
$$

MB et al. 2016, 2017, 2019; F. Fabis et al. 2018

# Some detail



- Generating functional:

$$Z[J] = \mathrm{e}^{\mathrm{i}\hat{S}_\mathrm{I}} \int \mathrm{d}\Gamma^{(\mathrm{i})} \mathrm{e}^{\mathrm{i}\int \langle J, \bar{x} \rangle \mathrm{d}t}$$

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$$\hat{S}_\mathrm{I} = \int \mathrm{d}1 \int \mathrm{d}2\, \hat{\rho}(1) \nabla v(12) \hat{\rho}(2)$$

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- Mean-field approximation: replace $\hat{S}_\mathrm{I}$ by $\langle S_\mathrm{I} \rangle = \hat{S}_\mathrm{I} Z[J]\big|_{J=0}$

# Specialisation to cosmology

1. Choose initial phase-space measure

$$\mathrm{d}\Gamma^{(\mathrm{i})} = P(q,p)\,\mathrm{d}^{3N}q\,\mathrm{d}^{3N}p$$

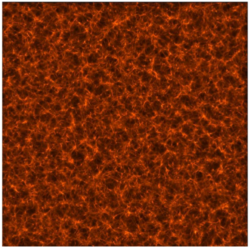
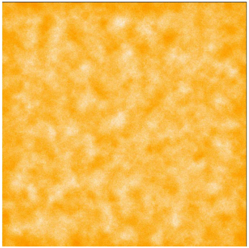
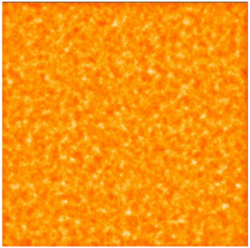
   fully specified by initial power spectrum
2. Change time coordinate

$$t \to \tau = D_+(t) - D_+(t_\mathrm{i})$$

3. Adapt Green's function to expanding universe

# Choice of Green's Function



MB 2015

# Interaction in the mean-field approximation



- Instead of perturbative expansion, replace interaction operator by its expectation value:

$$Z[J] = \int \mathrm{d}\Gamma^{(\mathrm{i})} \mathrm{e}^{\mathrm{i}\int \langle J, \bar{x}\rangle \mathrm{d}t}$$

with

$$\bar{x}(t) = G(t,0)x^{(\mathrm{i})} - \int_0^t G(t,t')\nabla v(t')\mathrm{d}t'$$

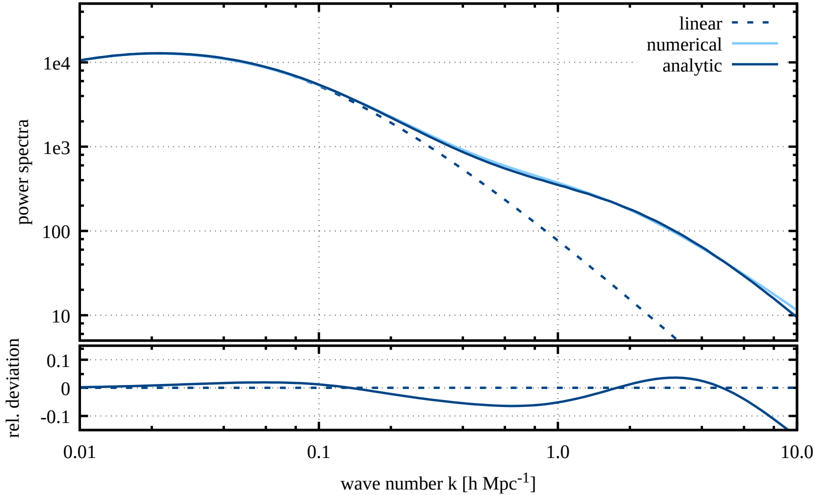
- Evaluate $\nabla v$ along inertial (unperturbed) trajectories
- Closed expression for non-linear power spectrum:

$$\bar{P}(k,\tau) = \mathrm{e}^{Q_\mathrm{D} + \mathrm{i}\langle S_\mathrm{I}\rangle} \int_q \left(\mathrm{e}^{\tau^2 k^2 a_\parallel(q)} - 1\right) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}}$$

MB et al. 2017; Dombrowski et al. 2018

# Density power spectrum

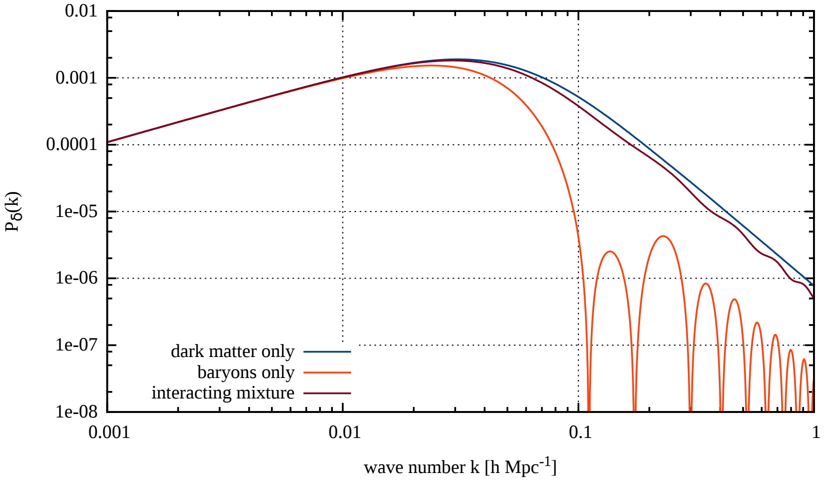





Mean-field approximation (MB et al. 2017, 2019, 2020)

# . . . including gas






I. Kostyuk, D. Geiss, F. Fabis, R. Lilow, C. Viermann 2016, 2018

# . . . with axions

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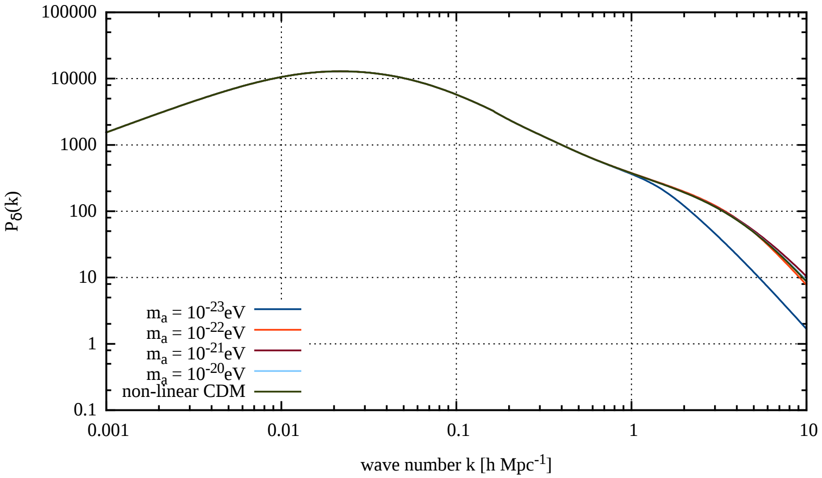
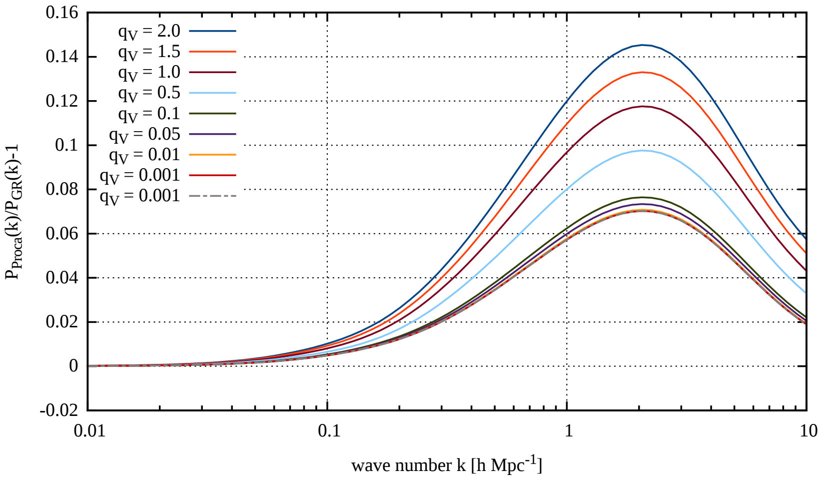



C. Littek et al. 2018

# . . . with an alternative field theory of gravity




L. Heisenberg, MB 2018

# Summary

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1. Non-equilibrium statistical field theory for dark-matter particles set up
2. Hamiltonian equations of motion, simple Green's function
3. Expansion parameter is deviation from unperturbed trajectories
4. $n$-point statistics for collective fields obtained by functional derivatives
5. Non-perturbative approach with Born approximation reproduces numerical results already quite well
6. Mixtures of gas and dark matter can be treated in the same way
7. Generalization to alternative matter models (axions)
8. Generalization to (some) alternative field theories of gravity straightforward

Recent review: Ann. Phys. 18 (2019) 00446