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Fermi Acceleration

Fermi's Original Proposal

Fermi (1949) proposed a model where particles can statistically gain energy by colliding with "moving magnetic fields". Possible ways of such "collisions" can either be magnetic mirror reflection (which Fermi called "type A"), or "type B" reflection where the particle simply follows the field line that bends backwards. The magnetic fields can be considered as being carried by interstellar clouds that move randomly with characteristic velocity $V \ll c$. The details of the interaction matters little, and the net effect is that the particle's velocity is reflected in the frame of the cloud. In reality, resonant interactions with random MHD waves/turbulence (which are moving!) are the most relevant process that scatter the particles.

Now let us estimate the energy gain/loss of the particle during a single collision. In doing so, we need to transform back and forth between the observer's frame, and the frame of the cloud. Quantities in the latter frame are denoted with prime $'$. Let the particle velocity, momentum and energy in the observer's frame be v, p and E (not independent, $p = Ev/c$). Let the cloud velocity be $-V$ along the x direction, $\gamma_V = (1 - V^2/c^2)^{-1/2}$ be the Lorentz factor of the cloud, and θ be the angle between the particle and the x axis. Using Lorentz transformation, particle energy and momentum in the cloud's frame is

$$E' = \gamma_V(E + Vp_x) \ , \quad p'_x = \gamma_V\left(p_x + \frac{VE}{c^2}\right) \ , \tag{1}$$

where $p_x = p\cos\theta$, $p'_x = p'\cos\theta'$. We use subscript $_1$ to denote quantities after the collision. In the cloud frame, we have $E'_1 = E'$, $p'_{x1} = -p'_x$. Transforming back to the observer's frame, we obtain

$$E_1 = \gamma_V(E' - Vp'_{x1}) = \gamma_V(E' + Vp'_x) = \gamma_V^2 E\left(1 + \frac{2Vv\cos\theta}{c^2} + \frac{V^2}{c^2}\right) \ . \tag{2}$$

Expanding to second order in V/c, we find

$$\frac{E_1 - E}{E} = \frac{\Delta E}{E} \approx \frac{2Vv\cos\theta}{c^2} + 2\frac{V^2}{c^2} \ . \tag{3}$$

Over time, the particle encounters many clouds whose velocities are oriented in random directions, and hence we need to average over θ. This average is not zero because there is a slightly higher probability for head-on collisions as opposed to trailing collisions. The probability $f(\theta)d\theta$ of a collision with angle between θ and $\theta + d\theta$ is proportional to the relative velocity $v + V\cos\theta$, and to the solid angle $d\Omega/4\pi \propto \sin\theta d\theta$ (where θ ranges from 0 to π). Let us treat the CR particle as being relativistic with $v \approx c$, then it is straightforward to see that

$$f(\theta)d\theta = \frac{1}{2}\left(1 + \frac{V\cos\theta}{c}\right)\sin\theta d\theta \ , \tag{4}$$

and hence

$$\left\langle \frac{2V\cos\theta}{c} \right\rangle = \frac{2V}{c}\int_0^\pi f(\theta)\cos\theta d\theta = \frac{2}{3}\frac{V^2}{c^2} \ . \tag{5}$$

Substituting it to (4), we see that by colliding with randomly moving clouds, the particle gains energy statistically, with mean energy gain at each collision given by

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{8}{3}\frac{V^2}{c^2} \ . \tag{6}$$

This is the well-known result of second-order Fermi acceleration. It is second order because energy gain is proportional to $(V/c)^2$. Obviously, the ultimate source of this energy gain comes from the kinetic energy of the clouds.

The second-order nature of this process has at least two disadvantages. First, it is too slow. Even around supernovae, we have $V/c \sim 10^{-4}$, and second-order processes are simply too inefficient to account for the presence of high energy cosmic-rays. In fact, the acceleration is too slow even to compete with ionization losses if one were to begin from low energies. Second, this process can in fact produce a power-law energy spectrum. However, the power-law index is unconstrained (determined by escape time / energy loss rate) and can presumably be any value, in contrast with observations that typically indicate a power law index between $2-3$.

On the other hand, second-order Fermi process may be responsible for the re-acceleration of existing population of non-thermal particles. For instance, radio halos in some galaxy clusters are likely produced by turbulent re-acceleration of mildly-relativistic electrons via the second-order Fermi process.

Making the Fermi Acceleration First Order: Diffusive Shock Acceleration

For many years, the second-order Fermi mechanism was considered the only possible way to accelerate particles, despite the difficulties associated with it. In the late 1970s, Blandford & Ostriker (1978) and Bell (1978) realized that Fermi acceleration can achieve first order around shock waves.

Let us consider a shock propagating in the x direction. We first work in the shock frame, where the plasmas enter the shock from upstream with velocity V_u from right to left, and are advected downstream with velocity V_d. Let r be the shock *compression ratio*. By mass conservation, we have

$$V_d = \frac{1}{r}V_u \ . \tag{7}$$

The velocity difference between the upstream and downstream flow is given by

$$V \equiv V_u - V_d = (r-1)V_d = \frac{r-1}{r}V_u \ . \tag{8}$$

Since $r > 1$, this means the plasma flow is *converging* across the shock front (in other words, $\nabla \cdot \boldsymbol{V} < 0$).

Consider CR particles in the vicinity of the shock front. These particles are largely collisionless and travel at velocities much larger than the shock velocity. Correspondingly, they barely feel (*directly*) the existence of the shock, but mainly respond to electromagnetic fluctuations in the plasma. We assume that both the shock upstream and downstream plasmas are sufficiently turbulent, so that the CR particles experience efficient scattering by such turbulence and get isotropized with respect to the upstream/downstream fluid that they reside. This means that every time a particle crosses the shock, it sees that the plasma is moving towards it. Performing a Lorentz transformation to the local frame of reference, the particle energy increases to the first order of V/c. The converging flow across the shock also facilitates the particle to be bounced back and forth many times, which is the essence for achieving the much more rapid first order acceleration. Because particle reflection is considered to be mediated by turbulent diffusion, this mechanism is also known as diffusive shock acceleration (DSA).

To calculate the rate of energy gain, let us again treat the CR particles to be relativistic with $v \sim c$, and assume the shock to be non-relativistic $V \ll c$. For first-order Fermi acceleration, it suffices to take the Lorentz factor $\gamma_V = 1$. At each reflection, the Lorentz transformation (1) becomes

$$E' = E + Vp\cos\theta \ , \qquad \text{or} \ \frac{\Delta E}{E} \approx \frac{V\cos\theta}{c} \ . \tag{9}$$

To calculate the mean energy gain at each reflection, we again need to average over θ similarly as we did before, but restrict the range of θ to $\cos\theta > 0$. Under our assumption that distribution of particles in the frame of background plasma (downstream or upstream) is isotropic, the number of particles within angles between θ and $\theta + d\theta$ is proportional to $\sin\theta d\theta$ (where θ ranges from 0 to $\pi/2$), and the rate they approach the shock front is $c\cos\theta$. Working out the normalization (a factor 2), the probability for crossing the shock at angle around θ is given by

$$f(\theta)d\theta = 2\cos\theta\sin\theta d\theta \ . \tag{10}$$

Therefore, the mean energy gain at every shock crossing is

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{V}{c}\int_0^{\pi/2} f(\theta)\cos\theta d\theta = \frac{2}{3}\frac{V}{c} \ . \tag{11}$$

Since particles are reflected twice per cycle (from upstream to downstream and then back to upstream), the mean energy gain per cycle is

$$\left\langle \frac{\Delta E}{E} \right\rangle_{\rm cycle} = \frac{4}{3}\frac{V}{c} \ . \tag{12}$$

Main Properties of Diffusive Shock Acceleration

Energy spectrum

Suppose that initially, there are N_0 particles with initial energy E_0. Let $E = AE_0$ be the mean particle energy after one cycle, and P be the probability that the particle remains in the acceleration region after one collision. Then, after k collisions, there are $N = N_0P^k$ particles with energy E_0A^k. In other words, after sufficient number of collisions, the *cumulative* particle energy distribution (number of particles with energy larger than E) becomes

$$N(\geq E) = N_0P^{\ln(E/E_0)/\ln A} = N_0\exp\left[\ln\frac{E}{E_0}\cdot\frac{\ln P}{\ln A}\right] = N_0\left(\frac{E}{E_0}\right)^{\ln P/\ln A} \ . \tag{13}$$

Its differential form is given by

$$f(E) = \frac{dN}{dE} \propto E^{-1+\ln P/\ln A} \ . \tag{14}$$

Clearly, we arrive at a power-law distribution $f(E) \propto E^{-s}$ with power law index

$$s = 1 - \frac{\ln P}{\ln A} \ . \tag{15}$$

Our previous analysis shows that $A = 1 + (4/3)V/c$. It remains to work out the escape probability P.

To estimate P, let us again work in the frame of the shock. Let J_+ be the flux of CR particles entering the shock from downstream, J_- be the flux of CR particles returning to the shock upstream from downstream, and J_∞ be the CR particle flux that escapes into the far downstream (towards infinity). Note that in the standard DSA, no particle escapes from the shock upstream because the upstream plasma is always advected towards the shock. In steady state, conservation of CR flux requires

$$J_+ = J_- + J_\infty \ . \tag{16}$$

The escape probability is thus given by

$$P = \frac{J_-}{J_+} = \frac{J_-}{J_- + J_\infty} \ . \tag{17}$$

Let n_0 be the CR particle number density in the vicinity of the shock front. The flux of particles crossing the shock surface from downstream is

$$J_- = \int_{\cos\theta > 0} \frac{d\Omega}{4\pi} nc\cos\theta = \frac{nc}{4} \ . \tag{18}$$

Since the CR particle distribution in the downstream frame is assumed to be isotropic, the escaping CR particle flux is simply given by

$$J_\infty = n_0 V_d \ . \tag{19}$$

Therefore, we obtain

$$P = \frac{c}{c + 4V_d} \approx 1 - \frac{4V_d}{c} \ . \tag{20}$$

Note that we assumed $V \ll c$, hence P is only slightly smaller than 1. In other words, only a tiny fraction of particles escape from the DSA process per cycle.

Jointly, the power law index of the CR energy spectrum resulting from DSA is given by

$$s = 1 - \frac{\ln P}{\ln A} \approx 1 + \frac{3V_d}{V} = 1 + \frac{3}{r-1} \ . \tag{21}$$

We see that in the DSA theory, a universal power-law index can be obtained which depends *only* on the shock compression ratio. For a strong non-relativistic shock (sonic and Alfvénic Mach number $\gg 1$), it is straightforward to show from shock jump conditions that the compression ratio $r = 4$. In this case, we arrive at $s = 2$. This is indeed close to the cosmic-ray energy spectrum (where $s \sim 2.7$), and the difference could be accounted for from propagation effects. It is also close though somewhat smaller than the index inferred from radio synchrotron spectrum of a variety of astrophysical sources (for electrons, $s \sim 2.2 - 3.0$).

Acceleration Rate and Maximum Energy

The rate of particle acceleration does depend on the more detailed microphysical processes in the shock upstream and downstream that scatter and isotropize the particles. Being a diffusive process, we simply assume that CRs spatially diffuse in the upstream/downstream with corresponding diffusion coefficients D_u and D_d.

Let us focus on the upstream first. In steady state, the CR density profile $n(x)$ is established from balancing advection towards the shock front by the upstream flow, and diffusion away from the shock by the CR density gradient

$$nV_u - D_u \frac{dn}{dx} = 0 \ . \tag{22}$$

For constant D_u, the solution is given by

$$n(x) = n_0 e^{-V_u x/D_u} \ , \tag{23}$$

with n_0 being the CR particle number density at the shock front. The total number of particles (actually column density) is

$$\int_{x=0}^{\infty} n(x)dx = \frac{n_0 D_u}{V_u}. \tag{24}$$

On the other hand, we just computed the particle flux entering into the shock upstream to be $J_- = n_0 c/4$. This means that the mean time a particle spends in the upstream is

$$t_u = \frac{N}{J_-} = \frac{4D_u}{V_u c} \ . \tag{25}$$

A similar argument applies to the downstream flow

$$t_d = \frac{4D_d}{V_d c} \ . \tag{26}$$

Therefore, the duration of one cycle is

$$t_{\rm cycle} = t_u + t_d = \frac{4D_u}{V_u c} + \frac{4D_d}{V_d c} \ . \tag{27}$$

This gives the rate of acceleration

$$\frac{dE}{dt} = \frac{4V}{3c}\frac{E}{t_{\rm cycle}} \equiv \frac{E}{t_a} \ , \tag{28}$$

where the acceleration time is

$$t_a = \frac{3ct_{\rm cycle}}{4V} = \frac{3}{V}\left(\frac{D_u}{V_u} + \frac{D_d}{V_d}\right) \ . \tag{29}$$

We see that the acceleration time depends on the efficiency of particle diffusion around the shock.

It remains to estimate the diffusion coefficients $D_{u,d}$. As usual, they can be written as

$$D \approx \frac{1}{3}\lambda v \ , \tag{30}$$

where λ is the effective particle mean free path, and v is particle velocity (in this case $v = c$). However, the value of D depends on the detailed microphysical processes, particularly on the level of turbulence in the shock. In the most optimistic case, known as the *Bohm limit*, we may take $\lambda \sim r_L$. The result is

$$D_{\rm Bohm} = \frac{1}{3} r_L c = \frac{Ec}{3ZeB} \ . \tag{31}$$

In reality, we expect $D \gtrsim D_{\rm Bohm}$, but the Bohm limit provides a very useful order-of-magnitude estimate.

Using the Bohm limit, we can estimate the acceleration time as

$$t_a = \frac{Ec}{Ve}\left(\frac{1}{B_u V_u} + \frac{1}{B_d V_d}\right) \equiv \alpha E \ . \tag{32}$$

This means that maximum particle energy increases linearly with time:

$$\frac{dE}{dt} = \frac{E}{t_a} = \frac{1}{\alpha} = \text{const} \ . \tag{33}$$

In the case of electrons, the maximum energy is typically limited by radiation losses, whose rate is typically proportional to E^2. Balancing acceleration rate and loss rate gives the maximum electron energy that can be achieved in shock acceleration.

In the case of ions, radiation loss is usually negligible, and the maximum energy is largely determined by the lifetime T of the accelerator. Taking $V_u = 4V_d$, $B_d \sim 4B_u$, we obtain

$$E_{\max} \approx \frac{3}{8} \frac{ZeB_u}{c} V_u^2 T \ . \tag{34}$$

For a supernova remnant shock, taking $t \sim 10^3$ years (the free expansion phase), $V_u \sim 5000$ km s^{-1}, $B_u \sim 10\mu$G (taking into account that it is pre-amplified from ISM field strength), we obtain a maximum energy of $E_{\max} \sim Z \times 0.8$ PeV, which is indeed comparable to the CR energy at the knee.