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Thermal Effects

Landau damping of Langmuir waves

To examine the complications associated with thermal effects, we restrict ourselves to an unmagnetized plasma, focusing on the Langmuir waves, or plasma oscillations. Here, it suffices to solve equations for the electrons only because ions barely move and mainly serve as a neutralizing background. The electron background state is trivial, described by a homogeneous distribution function $f_0 = f_0(\boldsymbol{v})$, with zero fields. With small perturbations, we can write the distribution function as

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\boldsymbol{v}) + f_1(\boldsymbol{r}, \boldsymbol{v}, t) \ . \tag{1}$$

Since background field is zero, the evolution of f_1 is simply governed by the linearized Vlasov equation, together with an equation that determines the electric field $\boldsymbol{E}_1$ (the Poisson equation)

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 - \frac{e}{m}\boldsymbol{E}_1 \cdot \nabla_{\boldsymbol{v}} f_0 = 0 \ , \tag{2}$$

$$\nabla \cdot \boldsymbol{E}_1 = -4\pi e \int f_1(\boldsymbol{x}, \boldsymbol{v}, t) d^3v \ . \tag{3}$$

Without loss of generality, we assume perturbations in the form of $e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, with $\boldsymbol{k} = k\boldsymbol{e}_x$. For electrostatic perturbations, $\nabla\times\boldsymbol{E}_1 = -(\partial\boldsymbol{B}/\partial t)/c = 0$, and hence $\boldsymbol{E}_1$ is also parallel to $\boldsymbol{k}$. The linearized Vlasov equation now becomes

$$-\mathrm{i}\omega f_1 + \mathrm{i}kv_x f_1 - \frac{e}{m}E_1\frac{\partial f_0}{\partial v_x} = 0 \ , \tag{4}$$

or

$$f_1 = \frac{\mathrm{i}}{\omega - kv_x}\frac{e}{m}E_1\frac{\partial f_0}{\partial v_x} \ . \tag{5}$$

Substituting the above to the Poisson equation, we obtain

$$kE_1 = -\frac{4\pi e^2}{m}E_1\int\frac{\partial f_0/\partial v_x}{\omega - kv_x}d^3v = -\frac{4\pi e^2}{m}E_1\int\frac{F_0'(v_x)}{\omega - kv_x}dv_x \ , \tag{6}$$

where $F_0(v_x) = \int f_0 dv_y dv_z$ is the reduced 1D electron distribution function.

More formally, the above equation can be reduced to $\epsilon E_1 = 0$, with dielectric function ϵ defined as

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{k^2 n_0}\int\frac{F_0'(v_x)}{v_x - \omega/k}dv_x \ . \tag{7}$$

The dispersion relation is simply given by $\epsilon = 0$.

How do we evaluate this integral? A mathematical difficulty arises at the singularity, $\omega = kv_x$. Clearly, this corresponds to a resonance where particle velocity equals to the phase velocity of the wave (called

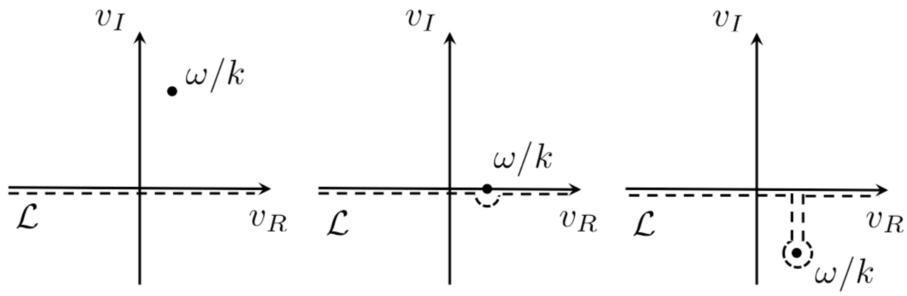

Figure 1: Illustration of the Landau contour $\mathcal{L}$ in the complex v plane.

Landau resonance). The resolution of this difficulty is owing to Landau (1946), which is described in detail in the textbook. Here we simply state the result.

To compute the dielectric function $\epsilon(k, \omega)$, we hold k fixed and evaluate it as a function of ω, treating ω as a complex number. While physically, v_x is real, mathematically, we can consider the integrand as a function of v_x in the complex plane, which has a pole at $v_x = \omega/k$. The integral is to be performed along the real axis, which is well defined when the pole is not at the real axis, either $\text{Im}(\omega) > 0$, or $\text{Im}(\omega) < 0$. However, there is a discontinuous jump in between, as the pole moves across the real axis in the velocity plane. Landau's analysis shows that causality considerations requires that $\epsilon(k, \omega)$ to be defined in the upper half velocity plane [$\text{Im}(\omega) > 0$] as in (8), and it is then analytically continued to the lower half plane. In doing so, the integration path must be kept to be below the pole ω/k, as the pole moves into the lower half velocity plane. This prescription of integration contour is called the *Landau prescription*, and the contour is called the *Landau contour*, denoted by $\mathcal{L}$, and is shown in Figure 1. The dispersion relation is then expressed as

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{k^2 n_0} \int_{\mathcal{L}} \frac{F_0'(v_x)}{v_x - \omega/k} dv_x = 0 \ . \tag{8}$$

In most situations, electrostatic waves are only weakly damped or unstable. We can write $\omega = \omega_R + \text{i}\omega_I$, with $|\omega_I| \ll \omega_R$, so that wave amplitude changes very little over one period. In this case, we can expand the dielectric function as follows

$$\epsilon(k, \omega_R + \text{i}\omega_I) \simeq \epsilon(k, \omega_R) + \text{i}\omega_I \frac{\partial \epsilon}{\partial \omega}\bigg|_{\omega=\omega_R} \simeq \epsilon_R(k, \omega_R) + \text{i}\left[\epsilon_I(k, \omega_R) + \omega_I \frac{\partial \epsilon_R}{\partial \omega_R}\bigg|_{\omega=\omega_R}\right] , \tag{9}$$

where in the second equality, we have used the fact that $|\omega_I| \ll \omega_R$ so that the $\omega_I \partial \epsilon_I / \partial \omega_R$ term is small compared with the ϵ_R term. This equation expresses the dielectric function slightly away from the real axis in terms of its value and derivative on and along the real axis. Specifically, the on-axis value can be obtained directly from the Landau contour

$$\epsilon(k, \omega_R) = 1 - \frac{\omega_p^2}{k^2 n_0} \mathcal{P} \int \frac{F_0'(v_x)}{v_x - \omega_R/k} dv_x - \text{i}\pi \frac{\omega_p^2}{k^2 n_0} F_0'(\omega_R/k) \ . \tag{10}$$

Here we introduce the Plemelj formula: for an analytical function $f(x)$ that is continuous along the real line

$$\lim_{\epsilon \to 0^+} \int \frac{f(x)}{x \pm \text{i}\epsilon} dx = \mp \text{i}\pi f(0) + \mathcal{P} \int \frac{f(x)}{x} dx \ , \tag{11}$$

where $\mathcal{P}$ demotes the Cauchy principle value

$$\mathcal{P}\int_a^b \frac{f(x)}{x}dx \equiv \lim_{\epsilon\to 0^+}\left[\int_a^{-\epsilon}\frac{f(x)}{x}dx + \int_\epsilon^b \frac{f(x)}{x}dx\right] . \tag{12}$$

The dispersion relation $\epsilon = 0$ requires that both the real and imaginary parts to vanish. For the real part, we have

$$1 - \frac{\omega_p^2}{k^2 n_0}\mathcal{P}\int \frac{F_0'(v_x)}{v_x - \omega_R/k}dv_x = 0 , \tag{13}$$

which determines ω_R. For the imaginary part, we have

$$\omega_I = -\frac{\pi F_0'(\omega_R/k)}{\frac{\partial}{\partial\omega_R}\mathcal{P}\int \frac{F_0'(v_x)}{v_x-\omega_R/k}dv_x} . \tag{14}$$

We see that wave damping/growth rate is proportional to the slope of the distribution function at the resonant velocity ω_R/k.

If we consider long-wavelength behavior ($v_{\rm ph} \approx \omega_p/k \gg v_{\rm th}$, or $k\lambda_D \ll 1$), we can expand the denominator as

$$\frac{1}{v_x - \omega_R/k} \approx -\frac{k}{\omega_R}\left[1 + \frac{kv_x}{\omega_R} + \left(\frac{kv_x}{\omega_R}\right)^2 + \left(\frac{kv_x}{\omega_R}\right)^3 + ...\right] . \tag{15}$$

Integrating by parts, the principle value component can be approximated by

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{F_0'(v_x)}{v_x - \omega_R/k}dv_x \approx \frac{k^2}{\omega_R^2}\int_{-\infty}^{\infty}F_0(v_x)\left[1 + 2\frac{kv_x}{\omega_R} + 3\left(\frac{kv_x}{\omega_R}\right)^2 + ...\right]dv_x . \tag{16}$$

Now consider a Maxwellian distribution

$$f_0(\boldsymbol{v}) = \frac{n_0}{(2\pi v_e^2)^{3/2}}\exp\left(-v^2/2v_e^2\right) , \quad F_0(v_x) = \frac{n_0}{\sqrt{2\pi}v_e}\exp\left(-v_x^2/2v_e^2\right) , \tag{17}$$

where $v_e = \sqrt{kT/m_e}$ is the electron thermal velocity. Plugging in to the above, we obtain

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{F_0'}{v_x - \omega_R/k}dv_x \approx n_0\frac{k^2}{\omega_R^2}\left(1 + \frac{3k^2}{\omega_R^2}v_e^2\right) , \quad F_0'(v) = -\frac{v}{v_e^2}F_0 . \tag{18}$$

The real part of the dispersion relation becomes

$$1 - \frac{\omega_p^2}{\omega_R^2}\left(1 + \frac{3k^2}{\omega_R^2}v_e^2\right) \approx 0 . \tag{19}$$

Since we have assumed that $\omega/k \gg v_e$, we see that to zeroth order, we simply have the dispersion relation for plasma oscillations $\omega_R^2 = \omega_p^2$. To the next order, we obtain

$$\omega_R^2 \approx \omega_p^2(1 + 3k^2\lambda_D^2) , \quad (k\lambda_D \ll 1) \tag{20}$$

We see that thermal effect makes Langmuir waves dispersive, with a group velocity of

$$v_g = \frac{d\omega_R}{dk} \approx 3k\lambda_D^2\omega_p = 3(k\lambda_D)v_e . \tag{21}$$

This is called the *Bohm-Gross* correction.

For the imaginary part, it is straightforward to obtain using (15)

$$\omega_I(k) = -\sqrt{\frac{\pi}{8}}\frac{\omega_p}{(k\lambda_D)^3}\exp\left[-\frac{1}{2(k\lambda_D)^2}-\frac{3}{2}\right] , \tag{22}$$

where we have used $\lambda_D = v_e/\omega_p$. Being negative, it means that Langmuir waves are damped. This collisionless damping phenomenon is called *Landau* damping. We see that the damping rate ω_I is strongly wavelength dependent, with $\omega_I \to 0$ for $k \to 0$, while damping rate approaches ω_p for $k\lambda_D \sim 1$ (at this point our approximation that $|\omega_I| \ll \omega_R$ already fails). Thus, plasma oscillations are strongly damped at Debye length scale. In other words, plasma oscillation can occur only at wavelength much larger than λ_D. This is easily understood because at the phase speed of the wave at Debye length scale is comparable to thermal speed, and are easily smeared out by thermal motions.

Physics of Landau damping

Mathematically, we have seen that Landau damping arises from the pole in the integral of ϵ at $v = \omega_R/k$, which corresponds to particles in resonance with the wave. Physically, this phenomenon first appears very puzzling: there is no dissipation in a collisionless system, yet waves are damped. This is due to energy exchange between the waves and particles.

In our conventional analysis of waves and instabilities, we perform a Fourier transform in both space and time, and analyze the wave modes in $(\boldsymbol{k}, \omega)$ space. The underlying assumption is that each wave mode makes particles oscillate around their unperturbed trajectories. Near the resonance, however, particles are in phase with the wave and hence feel a constant acceleration/deceleration. To show this, let us consider a test particle that travels very close to the phase speed of the wave. To the zeroth order, particle velocity is $v = v_0$, with $x = x_0 + v_0t$. To the next order we have

$$\frac{dv}{dt} = -\frac{e}{m}E_0e^{\mathrm{i}(kx_0+kv_0t-\omega t)} . \tag{23}$$

Solving this equation as an initial value problem, we find

$$v - v_0 = -\frac{e}{m}\left[\frac{e^{\mathrm{i}(kx_0+kv_0t-\omega t)} - e^{\mathrm{i}kx_0}}{\mathrm{i}(kv_0-\omega)}\right] . \tag{24}$$

Near the resonance as $kv_0 - \omega \to 0$, the above reduces to

$$v - v_0 \approx -\frac{e}{m}E_0te^{\mathrm{i}kx_0} , \tag{25}$$

showing that a particle with v_0 close to ω/k is constantly accelerated/decelerated with time, gaining or losing energy to the wave depending on its phase relative to the wave. For other particles that are non-resonant, their response to the wave is oscillatory without energy exchange with the wave.

To understand the net direction of energy transfer, more detailed consideration is needed. From the first order analysis above, we see that if we average over the spatial distribution of particles (assuming the background distribution is uniform), then the net work done by field is zero. In fact, we must go to the second order, calculate particle trajectories that are corrected by first-order velocities, and then average over particle initial positions. This will show that the particles traveling slightly faster than the wave gain energy in on average, while those traveling slightly slower than the wave lose energy on average. The net energy gain or loss by resonant particles thus depends on the slope of the distribution function at the resonance (see problem set).

Note that linear analysis fails for resonant particles, because their response are not oscillatory. The wave damping rate, on the other hand, is correctly captured by the mathematical trick following Landau's procedure.

Ion-acoustic waves

Our analysis of Landau damping above considered only the electron plasma. Now we add back in the ion component, and show that thermal effect introduces a new wave mode, the *ion acoustic* wave. We start from the dispersion relation, where it is straightforward to add in the contribution from the ions

$$\epsilon = 1 + \frac{\omega_{pe}^2}{kn_0}\int_{\mathcal{L}}\frac{F'_{0e}(v_x)}{\omega - kv_x}dv_x + \frac{\omega_{pi}^2}{kn_0}\int_{\mathcal{L}}\frac{F'_{0i}(v_x)}{\omega - kv_x}dv_x = 0\ . \tag{26}$$

We assume both electrons and protons follow an Maxwellian distribution, with temperatures T_e and T_i. Note that in general, the electron thermal velocity v_e is much larger than the ion thermal velocity v_i. For the time being, let us search for waves whose phase velocity satisfies $v_i \ll \omega/k \ll v_e$. Let us ignore Landau damping (the imaginary part) for the moment. In the ion term, we expand the denominator that includes the Bohm-Gross correction, while in the electron case, it suffices to just take $-(kv_x)^{-1}$. Note that for a Maxwellian distribution, $F'_{0e}(v_x) = -v_x/v_e^2 F_{0e}$. Now the dispersion relation reads

$$\epsilon = 1 + \frac{\omega_{pe}^2}{k^2v_e^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{k^2v_i^2}{\omega^2}\right) = 1 + \frac{1}{k^2\lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{k^2v_i^2}{\omega^2}\right) = 0\ . \tag{27}$$

The solution is

$$\omega^2 = \frac{k^2\lambda_D^2}{1 + k^2\lambda_D^2}\omega_{pi}^2 + 3k^2v_i^2 = \frac{T_e}{m_i}\frac{k^2}{1 + k^2\lambda_D^2} + 3k^2\frac{T_i}{m_i}\ . \tag{28}$$

The phase velocity is

$$v_{\rm ph} = \sqrt{\frac{T_e}{m_i}\frac{1}{1 + k^2\lambda_D^2} + 3\frac{T_i}{m_i}} \ll \sqrt{\frac{T_e}{m_e}} = v_e\ . \tag{29}$$

Thus our assumption $\omega/k \ll v_e$ is safely satisfied. Moreover, in order for $v_{\rm ph} \gg v_i$, we generally require $T_e \gg T_i$, otherwise, Landau damping on the ion plasma becomes significant (to be discussed shortly).

This dispersion relation looks very much like a sound wave, with pressure largely provided by thermal electrons (with $T_i \ll T_e$), and inertia provided by ions. As wavelength decreases towards the Debye length, wave frequency levels off and approaches the ion plasma frequency.

A major difference between ion acoustic wave and sound wave is the restoring force. In a sound wave, the pressure response has its origin from molecular collisions. For an ion acoustic wave, the restoring force has electrostatic origin and is due to the difference in electron and ion oscillations. Note that for Langmuir waves, the oscillation of electrons and ions are out of phase, while for ion-acoustic waves, their oscillations are in phase. Their amplitude ratio is simply given by $\sqrt{1 + k^2\lambda_D^2}$ (ion is larger). In the limit $k\lambda_D \gg 1$ (short wavelength or hot electrons), the oscillation is essentially ion plasma oscillation, with electrons playing a neutralizing background.

In the presence of finite ion temperature, the ion acoustic wave can severely suffer from Landau damping. In the long wavelength limit, since wave speed is much less than the electron thermal velocity, Landau damping due to electrons is always unimportant. For the ions, we rewrite the dispersion relation adding the imaginary part

$$1+\frac{1}{k^2\lambda_D^2}-\frac{\omega_{pi}^2}{\omega^2}\left(1+\frac{3k^2v_i^2}{\omega^2}\right)-\mathrm{i}\pi\frac{\omega_{pi}^2}{k^2n_0}F'_{0i}(\omega/k)=1+\frac{1}{k^2\lambda_D^2}-\frac{\omega_{pi}^2}{\omega^2}\left(1+\frac{3k^2v_i^2}{\omega^2}\right)+\mathrm{i}\pi\frac{\omega_{pi}^2\omega}{k^3v_i^2}\frac{F_{0i}(\omega/k)}{n_0}=0\ . \tag{30}$$

Let $\omega=\omega_R+\mathrm{i}\gamma$ and assuming $\gamma\ll\omega_R$, we obtain the imaginary part

$$\frac{\gamma}{\omega_R}\approx-\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{v_{\rm ph}}{v_i}\right)^3e^{-v_{\rm ph}^2/2v_i^2}\ . \tag{31}$$

We wee that Landau damping is very strong if $v_{\rm ph}$ is comparable to v_i. Thus the ion-acoustic wave can only propagate over a reasonable distance (more than a few wavelength) when T_e is at least a factor of $5-10$ times greater than T_i.

Ion-acoustic instability

We have seen that whether waves in a hot plasma can grow or damp depends on the sign of the velocity derivative of the distribution function at the resonant velocity. For a Maxwellian distribution, the result is always damping. However, if the distribution function has a bump sufficiently separated from the Maxwellian peak, one creates a positive slope, which may lead to its inverse: Landau growth.

A classical example is the ion-acoustic instability in a current-carrying system. A system that carries a current requires a drift velocity v_D between electrons and ions. Thus, the peak of Maxwellian distribution in the electrons and ions are shifted by v_D. If v_D exceeds the ion thermal speed v_i by a factor of a few, then at the phase speed of the ion acoustic wave, the electron distribution function would have a positive slope, which could potentially lead to Landau growth.

Let us work out the dispersion relation. Working in the ion-rest frame, then for all terms from the electron plasma in the dispersion relation, we replace ω by $\omega-kv_D$. To simplify, we assume that $v_D\ll v_e$ so that contribution of the electrons to the real part of the dispersion relation is largely unchanged, but now we need to take into account the imaginary part of the contribution. Working in the ion-rest frame, the dispersion relation reads

$$\begin{aligned}&1+\frac{1}{k^2\lambda_D^2}-\frac{\omega_{pi}^2}{\omega^2}\left(1+\frac{3k^2v_i^2}{\omega^2}\right)-\mathrm{i}\pi\frac{\omega_{pi}^2}{k^2n_0}F'_{0i}(\omega/k)-\mathrm{i}\pi\frac{\omega_{pe}^2}{k^2n_0}F'_{0e}(\omega/k-v_D)\\&=1+\frac{1}{k^2\lambda_D^2}-\frac{\omega_{pi}^2}{\omega^2}\left(1+\frac{3k^2v_i^2}{\omega^2}\right)+\mathrm{i}\pi\frac{\omega_{pi}^2\omega}{k^3v_i^2}\frac{F_{0i}(\omega/k)}{n_0}+\mathrm{i}\pi\frac{\omega_{pe}^2(\omega-kv_D)}{k^3v_e^2}\frac{F_{0e}(\omega/k-v_D)}{n_0}=0\ .\end{aligned} \tag{32}$$

To compare the contribution from ion Landau damping and electron current-driven Landau growth, it suffices to compare the relative strength between the two resonant terms. The overall result is

$$\frac{\gamma}{\omega_R}\approx-\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{v_{\rm ph}}{v_i}\right)^3\left[e^{-v_{\rm ph}^2/2v_i^2}+\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^{3/2}\left(1-\frac{kv_D}{\omega}\right)e^{-(v_{\rm ph}-v_D)^2/2v_e^2}\right]\ . \tag{33}$$

For our assumption $v_D\ll v_e$, the exponential term in the electron contribution can be neglected. We see that when v_D exceeds a threshold value

$$v_{\rm th} = v_{\rm ph}\left[1 + \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-v_{\rm ph}^2/2v_i^2}\right] . \tag{34}$$

Noting that at long-wavelength limit ($k\lambda_D \gg 1$),

$$\frac{v_{\rm ph}}{v_i} = \sqrt{\frac{T_e}{T_i} + 3} . \tag{35}$$

Therefore, the threshold drift velocity is sensitive to the temperature ratio.

Both the ion-acoustic instability and Buneman instability for a cold plasma involve drift motion between electrons and ions, and both are electrostatic in nature, but there is a fundamental difference. The Buneman instability derives from the real part of the dispersion relation and does not involve resonance. In this sense, it is a macroscopic fluid instability, where the entire fluid participates in it. On the other hand, the ion-acoustic instability is a *micro-instability*, which is driven by only a small fraction of particles in resonance with the wave. It is localized in velocity space, and most other particles are not directly involved so there is no bulk motion of the plasma as in fluid instabilities. However, such micro-instabilities can significantly affect the properties of a plasma. In particular, waves over a wide range of scales can be excited. Response of particles to these waves (or even turbulence!) effectively randomizes their motion, which appears as if they have undergone "collisions" even the plasma is collisionless. The outcome of such interactions can lead to anomalous or turbulent transport that can be much more effective than the classical collisional transport.

While we have derived separately the ion-acoustic instability and the Buneman instability in different limits of the parameter spaces. The behavior of the plasma transitions smoothly from the ion-acoustic instability to the Buneman instability as one increases v_D or decreases v_e.

The ion-acoustic instability has been attributed as a source of anomalous resistivity in current sheets in the context of magnetic reconnection. Because the waves generated by the instability extracts kinetic energy from the drifting electrons, and have the tendency to reduce their speeds. This effectively acts as a drag force to the electrons, corresponding to resistivity. Because no collisions are involved in the process, this is called anomalous resistivity. This topic will be discussed further later in the course.

Types of resonances

In this lecture, the resonance we have encountered is determined by

$$\omega - \boldsymbol{k}\cdot\boldsymbol{v} = 0 . \tag{36}$$

This is called the *Landau resonance*, after Landau who first predicted Landau damping due to this resonance. It is sometimes also called *Cherenkov resonance* because this resonance leads to the emission of Cherenkov radiation (particle travels at speed of light in a medium).

A second type of resonance is called the cyclotron resonance, which occurs in a magnetized plasma when

$$\omega - k_z v_z = \pm\Omega_L , \tag{37}$$

where Ω_L is the particle Larmor frequency, and the z axis is along $\boldsymbol{B}$. For parallel propagation, waves are either left or right circularly polarized. The resonance occurs only when the sense of particle gyration is the same as sense of electric vector rotation. Thus, electrons and ions are resonant with waves of different polarities. This is related to different resonance behaviors in left/right polarized cold plasma waves that we discussed in the previous lecture (in the special case of $v_z = 0$). The cyclotron resonance phenomenon is analogous to that of Landau damping, though the mathematics can be much more complex because we must deal with the full Vlasov-Maxwell equations.

The outcome of resonant interactions is two fold: it changes the amplitude of the wave, and it rearranges the distribution function in the resonant region. We have so far studied the first effect. The second is a non-linear process and will be studied in the next lecture.

The plasma dispersion function

Define *plasma dispersion function* $Z(\zeta)$

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - \zeta} dt \tag{38}$$

which is defined for $\mathrm{Im}(\zeta) > 0$, and is analytically continued for $\mathrm{Im}(\zeta) \leq 0$. It has a number of useful mathematical properties, and most notably, an alternative expression of $Z(\zeta)$ is given by

$$Z(\zeta) = \mathrm{i} e^{-\zeta^2} \int_{-\infty}^{\mathrm{i}\zeta} e^{-t^2} dt \ , \tag{39}$$

which makes it closely related to the error function, and it is within a constant factor to the well-studied *Dawson* function and *Faddeeva function*. There are standard mathematical routines that can quickly evaluate this function.

This function is often encountered in problems involving wave propagation through warm plasmas with Maxwellian velocity distribution. In our derivations of the dispersion relation of warm plasmas, we have effectively used the first few terms in the asymptotic expansion of $Z(\zeta)$. More details about the mathematical properties of this function can be found in the NRL plasma formulary.