Astronomy 253 (Spring 2016)

Xuening Bai

Jan. 25, 2016

Plasma Scales

Debye Shielding and Debye Length

Consider a background plasma of protons and electrons. It is neutral, homogeneous, with number density for both species being n_0, and has temperature T. There is no background electric or magnetic field. Now we insert a single, fixed test charge Q into the plasma and study how the plasma responds. We expect the mean density of electrons and protons to vary as smooth functions of radius r from the test charge, $n_e(r)$ and $n_p(r)$. The electrostatic potential $\Phi(r)$ outside of the test charge satisfies Poisson's equation

$$\nabla^2\Phi = -4\pi(n_p - n_e)e - 4\pi Q\delta(\boldsymbol{r}) \ . \tag{1}$$

The plasma is in thermal equilibrium, so that the density distribution follows the Boltzmann law. A proton at radius r from the test charge has an electrostatic potential energy $e\Phi(r)$. Correspondingly, the number densities are modified by the Boltzmann factor $\exp(e\Phi/k_BT)$ (assuming $\Phi = 0$ at $r = \infty$)

$$\begin{aligned} n_p &= n_0\exp(-e\Phi/k_BT) \approx n_0(1 - e\Phi/k_BT) \ , \\ n_e &= n_0\exp(+e\Phi/k_BT) \approx n_0(1 + e\Phi/k_BT) \ , \end{aligned} \tag{2}$$

where we have made a Taylor expansion of the Boltzmann, valid for $e\Phi \ll kB_T$. Inserting it back to the Poisson equation, we obtain

$$\nabla^2\Phi = \frac{8\pi n_0 e^2}{k_BT}\Phi - 4\pi Q\delta(\boldsymbol{r}) \ . \tag{3}$$

Note that without the plasma, the solution is simply $\Phi = Q/r$. The additional term in the above equation reflects shielding of the test charge by the plasma. The solution turns out to be

$$\Phi = \frac{Q}{r}\exp(-\sqrt{2}r/\lambda_D) \ , \tag{4}$$

where

$$\lambda_D \equiv \left(\frac{k_BT}{4\pi n_0 e^2}\right)^{1/2} = 6.9\left(\frac{T/K}{n_0/\mathrm{cm}^{-3}}\right)^{1/2} \mathrm{cm} \tag{5}$$

is called the Debye length. Thus, the test charge Q embedded in a plasma carries a cloud of particles with an excess of opposite charges, whose size is on the order of λ_D, and this cloud completely shields the test charge at distances $r \gtrsim \lambda_D$.

While we have assumed Q to be a test charge of external origin above, it can equally be a plasma electron/proton itself. One can consider each electron/ion as carrying its own cloud of opposite charges, and in the mean time, each of them contributes to the clouds of its neighboring particles. Typically, the Debye cloud contains huge number of particles, with almost the same number of electrons and ions. It is the tiny, time-averaged difference between the two that results in the exponential decay of the electrostatic potential.

We can perform a similar analysis on charge neutrality. Suppose there is a perturbation leading to an imbalance of charge density $\delta\rho = e(\delta n_p - \delta n_e)$. The resulting electrostatic potential satisfies

$$\nabla^2\Phi = -4\pi(\delta n_p - \delta n_e)e \ . \tag{6}$$

Of course, charge perturbation depends on scales. Let's say we are interested in scale L. By dimensional analysis, we have $\nabla^2\Phi \sim \delta\Phi/L^2$. Note that in weak field, we have $\delta n_e \sim n_0 e\delta\Phi/k_BT$. This leads to

$$\left|\frac{\delta n_p - \delta n_e}{\delta n_e}\right| \approx \frac{\delta\Phi/(4\pi e L^2)}{n_0 e\delta\Phi/kT} \approx \frac{\lambda_D^2}{L^2} \tag{7}$$

From this analysis, we can say that fluctuation in charge density is strong at scales $L \lesssim \lambda_D$, while the plasma is quasi-neutral at scales $L \gg \lambda_D$. We can thus consider λ_D as the electrostatic correlation length in a plasma.

Plasma Parameter and Collective Behavior

Note that in the above analysis of Debye shielding, we have implicitly assumed that there are many particles of opposite signs in the shielding cloud so that the electrostatic potential is a smooth function of r. The mean number of electrons in a Debye cloud is

$$N_D \equiv n_0\frac{4\pi}{3}\lambda_D^3 \ . \tag{8}$$

This is called the Debye number. A alternative definition that is often quoted is the plasma parameter

$$\Lambda \equiv 4\pi n_0\lambda_D^3 = \frac{(k_BT)^{3/2}}{(4\pi n_0)^{1/2}e^3} \approx 4.1\times10^3\frac{(T/1\ \mathrm{K})^{3/2}}{(n_0/\mathrm{cm}^{-3})^{1/2}} \ . \tag{9}$$

We now discuss the significance of this number, and show that when this number is large, inter-particle interactions are unimportant, leading to collective behavior.

To begin with, we introduce the following two length scales. First, the mean distance between particles is

$$r_d \approx n_0^{-1/3} \ . \tag{10}$$

Second, we consider the scale below which Coulomb interaction becomes important. This happens when the Coulomb potential energy, e^2/r, becomes comparable or stronger than particle kinetic energy, $\sim k_BT$. Equating the two leads to

$$r_c \approx \frac{e^2}{k_BT} \ . \tag{11}$$

The plasma parameter is closely related to the ratio of r_d/r_c:

$$\Lambda = \frac{\lambda_D}{(4\pi n_0\lambda_D^2)^{-1}} = \frac{\lambda_D}{(k_BT/2e^2)^{-1}} \approx \frac{\lambda_D}{r_c} \approx \frac{1}{\sqrt{4\pi}}\left(\frac{r_d}{r_c}\right)^{3/2} \ . \tag{12}$$

Now it becomes clear that if $r_d \ll r_c$ (or $\Lambda \ll 1$), kinetic energies of individual particles are small compared with the energy from Coulomb interactions. In other words, charged particles are dominated by one another's electrostatic influence. Such plasmas are considered *strongly coupled*. On the other hand, if $r_d \gg r_c$, strong Coulomb interactions between particles are relatively rare. Such plasmas are called *weakly coupled*. In this case, since $\Lambda \gg 1$, a large number of particles are responsible for a Debye cloud. This means that a tiny adjustment in particle distributions is sufficient to account for Debye shielding, which allows all particles to behave collectively (i.e., move together). It is this weakly coupled/collective regime that we shall consider in this course.

Plasma Oscillation, Plasma Frequency, and Skin Depth

Plasma oscillation, the relative oscillation of the plasma's electrons and ions, is the most fundamental phenomenon in plasma physics. It is also the most fundamental manifestation of plasma collective behavior. Consider a slab of plasma. Suppose for the moment the protons are all fixed and we displace all electrons rightward (in x) by a small amount ξ. This would create a net positive charge per unit area of $\sigma = en_0\xi$ on the left and the same amount of net negative charge on the right. This produces an electric field

$$E = 4\pi\sigma = 4\pi e n_0 \xi \ . \tag{13}$$

This electric field pulls on both the ions and the electrons in the plasma. But, because protons are much heavier, their acceleration is negligible compared with that of electrons, which we focus on. We have

$$\frac{d^2\xi}{dt^2} = -\frac{e}{m_e}E = -\frac{4\pi e^2 n_0}{m_e}\xi \ . \tag{14}$$

Since this a harmonic oscillator equation, the electrons oscillate sinusoidally with $\xi = \xi_0 \cos(\omega_{pe} t)$, at (electron) plasma frequency

$$\omega_{pe} \equiv \sqrt{\frac{4\pi e^2 n_0}{m_e}} = 5.64 \times 10^4 \left(\frac{n_0}{1 \ \mathrm{cm}^{-3}}\right)^{1/2} \mathrm{s}^{-1} \ . \tag{15}$$

Note that this frequency depends only on the plasma density n_0 but not on temperature or on the strength of magnetic field if present. Plasma oscillation is also called Langmuir waves, after *Irving Langmuir*. Note that if we define electron thermal speed

$$v_e = (k_B T/m_e)^{1/2} \ , \tag{16}$$

then we have

$$\omega_{pe} = v_e/\lambda_D \ . \tag{17}$$

In other words, a thermal electron travels about one Debye length in one electron plasma period. By the same token, we can consider ω_{pe}^{-1} as the electrostatic correlation time.

Plasma frequency depends on the mass of the charged particles. The electron plasma frequency, being the fastest and most fundamental, is generally just called the plasma frequency. There is the ion plasma frequency

$$\omega_{pi} \equiv \sqrt{\frac{4\pi e^2 n_0}{m_i}} = \sqrt{\frac{m_e}{m_i}}\omega_{pe} = 1.32 \times 10^3 \left(\frac{n_0}{1 \ \mathrm{cm}^{-3}}\right)^{1/2} \mathrm{s}^{-1} \ . \tag{18}$$

Similarly, a thermal ion travels one Debye length in one ion plasma period. The ion plasma frequency is also related to ion acoustic waves that will be covered later in the course.

Related to the plasma frequencies is another useful length scale, called the skin depth, or inertial length. The electron skin depth is defined as

$$d_e \equiv c/\omega_{pe} = 5.3\left(\frac{n_0}{1\ \mathrm{cm}^{-3}}\right)^{-1/2}\ \mathrm{km}\ . \tag{19}$$

Electromagnetic waves cannot propagate in a plasma at frequencies below ω_{pe}. In reality, such waves would be damped in just one plasma oscillation period, over which time the wave can propagate over a distance of d_e. It is a fundamental scale when we discuss electromagnetic waves in a plasma. Also note that $\lambda_D < d_e$ because $v_e < c$.

Analogously, there is the ion skin depth, also known as the ion inertial length, defined as

$$d_i \equiv c/\omega_{pi} = 230\left(\frac{n_0}{1\ \mathrm{cm}^{-3}}\right)^{-1/2}\ \mathrm{km}\ . \tag{20}$$

It characterizes the scale blow which ions become decoupled with the electrons and magnetic fields become frozen into the electron fluid rather than the bulk plasma. This will become clear when we discuss the generalized Ohm's law later in the course.